PHYSICS LETTERS A

Physics Letters A 303 (2002) 121–124

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# Generic incomparability of infinite-dimensional entangled states

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Received 11 May 2002; received in revised form 29 August 2002; accepted 29 August 2002

Communicated by P.R. Holland

In memory of Rob

**Abstract**

In support of a recent conjecture by Nielsen (1999), we prove that the phenomena of 'incomparable entanglement'—whereby, neither member of a pair of pure entangled states can be transformed into the other via local operations and classical communication (LOCC)—is a generic feature when the states at issue live in an infinite-dimensional Hilbert space.


*PACS:* 03.65.Ud; 03.65.Ta; 03.67.-a

*Keywords:* Entanglement; Local operations; Classical communication; Majorization; Incomparable states

## Nielsen's characterization theorem and conjecture

Let $\mathcal{H}_n$ be a Hilbert space of countable dimension $n \geqslant 2$. For unit vectors $\psi_{1,2} \in \mathcal{H}_n \otimes \mathcal{H}_n$, i.e., two states of a composite system with two isomorphic subsystems, let $\psi_1 \prec \psi_2$ denote that it is possible to transform $\psi_1$ into $\psi_2$ with certainty by performing local operations on the subsystems and communicating classically between their locations (LOCC). (See [1], Section 12.5.1, for a complete discussion.) Let $\rho_{\psi_i}$ denote the reduced density operator on $\mathcal{H}_n$ determined by the state $\psi_i$, and let $\vec{\rho}_{\psi_i} = \{\lambda_i^{(1)}, \ldots, \lambda_i^{(n)}\}$ denote the vector of $\rho_{\psi_i}$'s eigenvalues, i.e., $\psi_i$'s squared Schmidt coefficients, arranged in non-increasing order. Then Nielsen's [2] characterization theorem asserts that $\psi_1 \prec \psi_2$ iff $\vec{\rho}_{\psi_1}$ is *majorized* by $\vec{\rho}_{\psi_2}$, i.e., iff for all $k = 1, \ldots, n$,

$$\sum_{j=1}^{k} \lambda_1^{(j)} \leqslant \sum_{j=1}^{k} \lambda_2^{(j)}.$$

One corollary of this elegant little characterization is the following simple result, to be used later on. The Schmidt number, $\sharp\psi_i$, of a state $\psi_i$ is defined to be the number of nonzero entries of the vector $\vec{\rho}_{\psi_i}$. Thus, Nielsen's theorem makes it easy to see that a state's Schmidt number cannot be increased under LOCC; for, if $\sharp\psi_1 < \sharp\psi_2$, then the $\sharp\psi_1$-th inequality in the majorization condition must necessarily fail due to the normalization of the eigenvalues of a reduced density operator. In particular, then, it follows that a product

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0375-9601/02/$ – see front matter 
PII: S0375-9601(02)01228-8

state, for which the Schmidt number is 1, cannot be LOCC-transformed into an entangled state—which, of course, we already know must be true, because entanglement between systems cannot be created by local operations on either of them alone.

Consider, now, the set, $S_{\text{inc}}$, of all pairs $(\psi_1, \psi_2)$ such that $\psi_1 \not\prec \psi_2$ *and* $\psi_2 \not\prec \psi_1$, where 'inc' stands for *incomparable*, in Nielsen's [2] terminology. In the same paper (cf. [2], p. 3), Nielsen gave a heuristic argument for the claim that the probability of picking at random two incomparable states out of the set of all $n \times n$ entangled pure states—according to the natural, rotationally invariant measure—tends to 1 as $n \to \infty$. If true, this conjecture would appear to establish that there is a large variety of different non-interconvertible forms of pure state entanglement encountered as the dimension of a system's state space increases without bound. However, it is not obvious how to complete Nielsen's reasoning with a simple but rigorous argument; e.g., Życkowski and Bengtsson (see [3], Section IIID) have given another argument based upon geometrical considerations, but it too is no more than heuristic.

So in this Letter, we shall focus on rigorously establishing an elementary but slightly different result that equally well supports the intuition that the complexity of pure state entanglement increases with dimension: namely, when $n = \infty$, the set of pairs in $S_{\text{inc}}$ lie open and dense in the Cartesian product of the unit sphere of $\mathcal{H}_n \otimes \mathcal{H}_n$ with itself. Here, the physically relevant topology is that induced by the standard Hilbert space norm, which, in particular, guarantees that two pairs of pure states will qualify as close only if they (pairwise) dictate *uniformly* close expectation values for all observables. Due to the fact that the unit sphere of an infinite-dimensional Hilbert space is not even locally compact, there is no sensible Lebesgue-type measure on the set of pairs of unit vectors taken from $\mathcal{H}_\infty \otimes \mathcal{H}_\infty$ (cf. [4], p. 241). Thus, the statement that $S_{\text{inc}}$ is norm open and dense in the infinite-dimensional case is the strongest statement about the genericity of the set of incomparable states that one can possibly hope to make as the complement of $S_{\text{inc}}$ is then 'nowhere dense' and so has measure zero.

To put it more plainly, genericity, in this context, amounts to firstly that within any finite region of the $\mathcal{H}_\infty \otimes \mathcal{H}_\infty$ space of state pairs there are uncountably many pairs that are incomparable. Secondly, since the set of comparable pairs is closed (as it's the complement of the open set $S_{\text{inc}}$) then there are comparable states on the boundary of the set for which an approximating incomparable state can be found as close as you like. The converse cannot be said of any incomparable state. In this sense we claim that incomparability is more common than comparability and hence that the former is a generic property.

Moreover, as will be seen, our method of proof actually establishes that densely many of these generically incomparable pairs are in fact *strongly* incomparable in the sense of Bandyopadhyay et al. [5]: i.e., they cannot even be converted into one another with the help of an entanglement catalyst, or by performing collective local operations on multiple copies of the input state. Thus our result actually *strengthens* the intuition behind Nielsen's conjecture.

## Proof of generic incomparability for infinite-dimensional states

Let us first establish that $S_{\text{inc}}$ is open—or, equivalently, its complement $S'_{\text{inc}}$ is closed—which happens to be true for *any* countable value of $n$. To this end, let us write, when $k$ is finite, '$\psi_1 \prec_k \psi_2$' just in case

$$\sum_{j=1}^{k} \left(\lambda_1^{(j)} - \lambda_2^{(j)}\right) \leqslant 0;$$

and let us define $S_{\prec_k} \equiv \{(\psi_1, \psi_2)\colon\ \psi_1 \prec_k \psi_2\}$, with a similar definition for $S_{\succ_k}$. By Nielsen's theorem,

$$S'_{\text{inc}} = \left(\bigcap_{k=1}^{\infty} S_{\prec_k}\right) \cup \left(\bigcap_{k=1}^{\infty} S_{\succ_k}\right),$$

and so it suffices for us to show that each $S_{\prec_k}$ is closed (the argument for each $S_{\succ_k}$ being closed is the same, by symmetry).

First, note that the mappings $\psi_i \mapsto \rho_{\psi_i}$ and $\rho_{\psi_i} \mapsto \{\lambda_i^{(1)}, \ldots, \lambda_i^{(k)}\}$ are both trace-norm continuous (see [6], Eqs. (1)–(5)), and it is easy to see that the mapping from $\mathbb{R}^k \times \mathbb{R}^k$ to $\mathbb{R}$:

$$\left(\{\lambda_1^{(1)}, \ldots, \lambda_1^{(k)}\},\ \{\lambda_2^{(1)}, \ldots, \lambda_2^{(k)}\}\right) \mapsto \sum_{j=1}^{k} \left(\lambda_1^{(j)} - \lambda_2^{(j)}\right)$$

is jointly continuous. Therefore, so too is the mapping defined by

$$\Phi(\psi_1,\psi_2)=\sum_{j=1}^{k}\bigl(\lambda_1^{(j)}-\lambda_2^{(j)}\bigr).$$

Now, let $(\psi_{1m},\psi_{2m})\in S_{\prec_k}$ be any Cauchy sequence, where, by the completeness of Hilbert space, we know there exists a limit pair $(\tilde{\psi}_1,\tilde{\psi}_2)$. To show that $S_{\prec_k}$ is closed, we must show that it also contains this limit pair. But, recalling that $\Phi$ is continuous, we know that

$$\bigl\{\Phi(\psi_{1m},\psi_{2m})\bigr\}=\left\{\sum_{j=1}^{k}\bigl(\lambda_{1m}^{(j)}-\lambda_{2m}^{(j)}\bigr)\right\}$$

must be a Cauchy sequence too; and, since the nonpositive real numbers are closed, it follows that this latter sequence converges to a real number

$$\sum_{j=1}^{k}\bigl(\tilde{\lambda}_1^{(j)}-\tilde{\lambda}_2^{(j)}\bigr)\leqslant 0.$$

Thus $(\tilde{\psi}_1,\tilde{\psi}_2)\in S_{\prec_k}$, as required.

Turning now to the density of $S_{\text{inc}}$, what we shall actually establish is that the set of *strongly incomparable* pairs, $S_{\text{st inc}}$, is dense when $n=\infty$. A pair of entangled states $(\psi_1,\psi_2)$ is called *strongly* incomparable just in case $\psi_1\not\prec\psi_2$ *and* $\psi_2\not\prec\psi_1$ *and* it is not possible to convert finitely many copies of one of $(\psi_1,\psi_2)$ into the other, even with the help of a (finite-dimensional) catalyst. To say that $\psi_1$ cannot be converted into $\psi_2$, even using multiple copies, is to say that for *no* (finite) value of $m$ is it the case that

$$\underbrace{\psi_1\otimes\cdots\otimes\psi_1}_{m\text{ times}}\prec\underbrace{\psi_2\otimes\cdots\otimes\psi_2}_{m\text{ times}}.$$

That there are states that cannot be transformed *singly* into each other by LOCC, but *can* be so transformed by local collective operations if multiple copies of the input state are available, was confirmed recently by Bandyopadhyay et al. [5]. To say that a state $\psi_1$ cannot be converted into $\psi_2$ even with the help of a catalyst is simply to say that there is *no* entangled state $v$ (with finite Schmidt number) such that $\psi_1\otimes v\prec\psi_2\otimes v$. Again, it was pointed out by Jonathan and Plenio [7] that there are states that cannot be transformed into each other by LOCC, but *can* be so transformed with the help of a suitable catalyst.

Henceforth, we shall require only one simple *sufficient* condition for a pair of states $(\psi_1,\psi_2)$ with finite Schmidt numbers to be strongly incomparable, viz.,

$$\begin{array}{c}\lambda_1^{(1)}>\lambda_2^{(1)}\quad\text{AND}\quad\sharp\psi_1>\sharp\psi_2,\\ \text{OR}\\ \lambda_1^{(1)}<\lambda_2^{(1)}\quad\text{AND}\quad\sharp\psi_1<\sharp\psi_2.\end{array}\tag{C}$$

Let us prove by reductio that this condition, (C), is indeed sufficient for incomparability. Thus, suppose that, in fact, $(\psi_1,\psi_2)$ are *not* strongly incomparable, but that their respective Schmidt values meet condition (C). Then, either for some finite $m$ there is a catalyst $v$ such that $\psi_1^{\otimes m}\otimes v\prec\psi_2^{\otimes m}\otimes v$ or, similarly, in the reverse direction. But then, by the first majorization condition in Nielsen's characterization theorem, plus its corollary that a state's Schmidt number cannot be increased under LOCC, it follows that either

$$\bigl(\lambda_1^{(1)}\bigr)^m\lambda_v^{(1)}\leqslant\bigl(\lambda_2^{(1)}\bigr)^m\lambda_v^{(1)}\quad\text{AND}$$
$$(\sharp\psi_1)^m(\sharp v)\geqslant(\sharp\psi_2)^m(\sharp v),$$

or that the same two expressions hold with the inequalities reversed. Thus, upon cancellation, we see that it must be the case that either $\lambda_1^{(1)}\leqslant\lambda_2^{(1)}$ and $\sharp\psi_1\geqslant\sharp\psi_2$, or $\lambda_1^{(1)}\geqslant\lambda_2^{(1)}$ and $\sharp\psi_1\leqslant\sharp\psi_2$—a condition that is easily seen to be logically inconsistent with (C).

Turning, finally, to the proof that $S_{\text{st inc}}$ is dense, first observe that the set of all $\psi\in\mathcal{H}_\infty\otimes\mathcal{H}_\infty$ for which the entries of $\vec{\rho}_\psi$ are all nonzero—for simplicity, we call these *complete* states—is itself dense. For, if a state $\psi$ merely has a Schmidt decomposition involving finitely many terms, i.e.,

$$\psi=\sum_{j=1}^{p<\infty}\sqrt{\lambda^{(j)}}\,x^{(j)}\otimes y^{(j)},$$

it is approximated arbitrarily closely by the sequence of (normalized) complete states

$$\psi_m=\sum_{j=1}^{\infty}\sqrt{\tilde{\lambda}^{(j)}}\,\tilde{x}^{(j)}\otimes\tilde{y}^{(j)},$$

where

$$\tilde{\lambda}^{(j)}\equiv\begin{cases}\lambda^{(j)}/\bigl(1+m^{-1}\bigr)>0 & \text{when } j\leqslant p,\\ 1/\bigl(2^{j-p}(m+1)\bigr)>0 & \text{when } j>p,\end{cases}$$

and the orthonormal *bases* $\tilde{x}^{(j)}$, $\tilde{y}^{(j)}$ respectively extend the orthonormal sets $x^{(j)}$, $y^{(j)}$ beyond the index value $p$. Thus, the set consisting of complete pairs of states $(\psi_1, \psi_2)$ is a dense set. Furthermore, it is quite easy to see that any complete pair of states with $\lambda_1^{(1)} = \lambda_2^{(1)}$ can be arbitrarily closely approximated by complete pairs that do *not* satisfy that identity. So, in sum: the set, call it $S_{c\neq}$, of all complete pairs of states, whose first Schmidt coefficients are *un*equal, form a dense set. We are going to show that every element of $S_{c\neq}$ can itself be approximated arbitrarily closely using members of $S_{\text{st inc}}$, which therefore must *also* be a dense set.

So let $(\psi_1, \psi_2) \in S_{c\neq}$ be arbitrary. If $\lambda_1^{(1)} > \lambda_2^{(1)}$, let us choose the sequence of (finite Schmidt number) pairs $(\psi_{1m}, \psi_{2m})$ in such a way that

$$\vec{\rho}_{\psi_{1m}} = \left\{ \frac{\lambda_1^{(1)}}{\sum_{j=1}^m \lambda_1^{(j)}}, \ldots, \frac{\lambda_1^{(m)}}{\sum_{j=1}^m \lambda_1^{(j)}}, 0, 0, 0, \ldots \right\},$$

$$\vec{\rho}_{\psi_{2m}} = \left\{ \frac{\lambda_2^{(1)}}{\sum_{j=1}^{m-1} \lambda_2^{(j)}}, \ldots, \frac{\lambda_2^{(m-1)}}{\sum_{j=1}^{m-1} \lambda_2^{(j)}}, 0, 0, 0, 0, \ldots \right\}.$$

By construction, $\lim_{m\to\infty}(\psi_{1m}, \psi_{2m}) = (\psi_1, \psi_2)$, $\sharp\psi_{1m} > \sharp\psi_{2m}$ for all $m$ (since, by completeness of $(\psi_1, \psi_2)$, $\lambda_1^{(j)}, \lambda_2^{(j)} \neq 0$ for all $j$), and *for all sufficiently large* $m$, $\lambda_{1m}^{(1)} > \lambda_{2m}^{(1)}$ (since, $\lambda_1^{(1)} > \lambda_2^{(1)}$). Thus, the pairs $(\psi_{1m}, \psi_{2m})$ approximate $(\psi_1, \psi_2)$ and, in virtue of satisfying condition (C), are strongly incomparable for all sufficiently large $m$. Similarly, if instead $\lambda_1^{(1)} < \lambda_2^{(1)}$ holds for the pair $(\psi_1, \psi_2)$, an analogous approximating sequence of (for all sufficiently large $m$, strongly incomparable) pairs $(\psi_{1m}, \psi_{2m})$ is obtained simply by interchanging the definitions of $\psi_{1m}$ and $\psi_{2m}$ above.

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