Structure, formally speaking — Phil 246

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Remark. I claim no originality in the following content, and sometimes not even for the particular formulations used. There are only so many ways in which one can express mathematical definitions grammatically and with full rigour. In order not to break the flow of the presentation, however, I have not given precise credit where it is due, but only summarily in the references.

1 Preliminaries

"Structure" is the central concept around which this seminar is organized. Before we look into its particular conceptualization and the role it plays in various areas of philosophy, such as metaphysics, epistemology, and philosophy of science, we want to get a quick and dirty introduction into the way mathematics and its various subfields has thought of structure.

Before I do that, I would like to state that the usual, typically tacit, way philosophers think of a "structure" is a simplified version of what will be referred to below as a *relational structure*, i.e. the set-theoretic notion found in mathematical logic according to which a structure $\mathcal{S}$ consists of an ordered pair $\langle \mathcal{O}, \mathcal{R} \rangle$ which consists of a non-empty set of relations $\mathcal{R}$ ("ideology") as well as a non-empty set of relata $\mathcal{O}$ ("ontology"), the *domain* of $\mathcal{S}$. But I am getting ahead of myself. Just a final warning. This standard notion has recently been challenged by various authors. Leitgeb and Ladyman (2008) explore a graph-theoretic notion of structure, and Roberts (forthcoming) proposes a group-theoretic one. Landry (2007) argues for conceptual pluralism in that a restriction to the standard notion (or any *one* notion, for that matter) cannot do justice to the plethora of applications. Muller (forthcoming) makes the case that a thoroughgoing structuralism should not rely on a conceptualization of structure that rests on a definition in terms of more basic concepts. Instead, he argues, structure should be given as a primitive, encoded in an axiomatization.

2 Structures, informally speaking

Informally speaking, a "structure" is defined on a set (e.g. of objects) as additional mathematical objects associated with the set. These additional objects defined on a set may be measures, algebraic structures (groups, fields, etc.), topologies, metric structures (geometries), orders, equivalence relations, and differential structures.

Wikipedia gives examples of structure defined on the set $\mathbb{R}$ of the real numbers (entry: "Mathematical structures"):

- an *order*: each number is either less or more than every other number

- *algebraic structure*: there are operations of multiplication and addition that make it into a field

- a *measure*: intervals along the real line have a certain length, which can be extended to the Lebesgue measure on many of its subsets

- a *metric*: there is a notion of distance between points

- a *geometry*: it is equipped with a metric and is flat

- a *topology*: there is a notion of open sets

Not all of these structures are independent: e.g., its order induces a topology, as does, independently, its metric structure. Two structures may combine to yield another structure: e.g., its order and algebraic structure create an ordered field, and its algebraic structure and its topology combine to give the structure of a Lie group.

Structures are often characterized by the mappings between sets which preserve them, in the sense that structures defined on the domain and those defined on the range (or "codomain") are equivalent (in some sense). Roughly, an "homomorphism" is a "structure-preserving" map from one algebraic structure (e.g., ring, group, vector space) to another. Similarly, a "homeomorphism" (note the difference!) preserves topological structures and a "diffeomorphism" preserves differential structures. If a bijective homomorphism has an inverse that is also a homomorphism, then we say it's a "isomorphism". Isomorphisms are thus often used to capture structural equivalence: two structures $\mathcal{A}$ and $\mathcal{B}$ are *structurally identical* just in case they are isomorphic, denoted $\mathcal{A} \simeq \mathcal{B}$, i.e. iff there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. An isomorphism from a set onto itself is an *automorphism*.

3 Let's get mathematical

Let's try to express some of this more rigorously. In most mathematical disciplines, particularly in mathematical logic and model theory, a structure is more or less defined as follows (although there exists a certain variation). A structure consists of a set of elements together with a collection of finitary functions and relations defined on the set. More precisely, a *structure* (or *τ-structure*) $\mathcal{A}$ is an ordered triple $\langle A, \tau, I \rangle$ consisting of a domain (or universe) A, a signature τ, and an interpretation function I of τ. The *domain A* of $\mathcal{A}$, sometimes also denoted $\mathrm{dom}(\mathcal{A})$ (or $|\mathcal{A}|$, but that's unfortunate, as we will use this to denote the *order* of a structure of a graph), is just an arbitrary, usually non-empty, set.

3.1 Signatures...

The *signature τ* of a structure is an ordered triple $\langle R_n, F_n, \mathrm{ar} \rangle$ consisting of a (usually countable) set R_n of n-ary relation symbols, a (usually countable) set F_n of n-ary function symbols, and a function $\mathrm{ar} : R_n \cup F_n \to \mathbb{N}_0$ which assigns a natural number (including 0) called *arity* to every symbol in $R_n \cup F_n$. The sets R_n and F_n are required to be disjoint. A symbol in $R_n \cup F_n$ is called *n-ary* if it arity is n. A 0-ary function symbol is called a constant symbol. A signature without any function symbol is called a *relational signature*, and one without any relation symbol an *algebraic signature*. Structures with relational (algebraic) signatures are called *relational (algebraic) structures*.

A function f is *n-ary* just in case there exist sets $X_1, ..., X_n, Y$ such that $f : X_1 \times \cdots \times X_n \to Y$, where $X_1 \times \cdots \times X_n$ is the Cartesian product of $X_1, ..., X_n$. The Cartesian product $X_1 \times \cdots \times X_n$ of $X_1, ..., X_n$ can be defined as the set of all ordered n-tuples $\langle x_1, ..., x_n \rangle$ such that for all $i = 1, ..., n$, $x_i \in X_i$. An *n-ary relation* defined on sets $X_1, ..., X_n$ is a set of ordered n-tuples $\langle x_1, ..., x_n \rangle$, where $x_i \in X_i$ for all $i = 1, ..., n$. Thus, an n-ary relation on sets $X_1, ..., X_n$ is just a subset of the Cartesian product $X_1 \times \cdots \times X_n$ of these sets. Elements in the set of constant symbols symbolize non-logical constants than name individuals.

This is terribly abstract. Examples of n-ary function symbols are $+$ and $\times$ (both have $n = 2$), examples of n-ary relation symbols are $\leq$ and $\in$ (again, in both cases $n = 2$), examples of constant symbols are 0 and 1. Here are some examples of signatures:

- A signature could simply consist of a binary relation symbol $<$, a binary function symbol $\times$, and a constant (0-ary function) symbol 1.

- A signature of sets is the symbol $\emptyset$ denoting the empty set.

- A signature of groups is a set $\{e, ^{-1}, \times\}$, where e is the constant symbol denoting the group identity, $^{-1}$ is an unary function symbol denoting the group inverse operation, and $\times$ is a binary function symbol denoting the group multiplication.

- A signature of posets (partially ordered sets) is a singleton $\{\leq\}$, where $\leq$ is a binary relation symbol denoting the partial ordering relation.

Remark (from PlanetMath). Given a signature τ, the set L of logical symbols (from first-order logic), and a (countably infinite) set V of variables, we can form a first-order language, consisting of all formulas built from these symbols (in $\tau \cup L \cup V$). The resulting language is uniquely determined by τ.

3.2 ... and their interpretation

The *interpretation function I* of τ assigns functions (and constants) and relations to the symbols of τ. This means that each function symbol f of arity n is assigned an n-ary function on the domain and each relation symbol R of arity n is assigned an n-ary relation on the domain. The interpretation of constant symbols assigns them a constant element in the domain. Often, in given contexts, no notational distinction is made between a symbol σ and its interpretation $I(\sigma)$.

3.3 An example of a structure: rational numbers

An important class of structures are *algebras* defined over fields. You have all come across algebras, although you may not have realized it. The domain of an algebra (which is something more specific than mere algebraic structures) is a field,[1] such as the field of rational numbers $\mathbb{Q}$, of real numbers $\mathbb{R}$, or of complex numbers $\mathbb{C}$. The standard signature τ_f for fields is a set consisting of two binary function symbols $+$ and $\times$, a unary function symbol $-$, and two constant symbols 0 and 1. A structure (in this case, an algebra) for a signature of this type thus consists of a set of elements, with two binary functions, a unary function, and two distinguished elements, although it doesn't have to satisfy the field axioms (such as associativity and commutativity for addition and multiplication). Consider, for instance, the field of rational numbers $\mathbb{Q}$, which can be regarded as a τ_f-structure in a straightforward way:

$$\mathcal{Q} = \langle \mathbb{Q}, \tau_f, I_{\mathcal{Q}} \rangle, \tag{1}$$

such that $I_{\mathcal{Q}}(+) : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is addition of rational numbers, $I_{\mathcal{Q}}(\times) : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is multiplication of rational numbers, $I_{\mathcal{Q}}(-) : \mathbb{Q} \to \mathbb{Q}$ is the unary function which takes every rational number q to $-q$, $I_{\mathcal{Q}}(0) \in \mathbb{Q}$ is the number 0, and $I_{\mathcal{Q}}(1) \in \mathbb{Q}$ is the number 1.

3.4 Morphisms

Given two structures $\mathcal{A}$ and $\mathcal{B}$ with the same signature τ, a *(τ-)homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a map $h : A \to B$ which preserves the functions and relations. "Preserving functions and relations" is to be understood as follows:

- For any n-ary function symbol f in τ and any elements $a_1, ..., a_n \in A$, $h(f(a_1, ..., a_n)) = f(h(a_1), ..., h(a_n))$.

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[1] A *field* $\mathbb{F}$ is a set on which two operations, usually "addition" and "multiplication," are defined such that the following axioms obtain: closure of $\mathbb{F}$ under addition and multiplication, associativity and commutativity of addition and multiplication, additive and multiplicative identity and inverses, and distributivity of multiplication over addition.

- For any n-ary relation symbol R in τ and any elements $a_1, ..., a_n \in A$, $(a_1, ..., a_n) \in R \Rightarrow (h(a_1), ..., h(a_n)) \in R$.

A typical problem that mathematicians are interested in is the *homomorphism problem*: given two finite relational structures $\mathcal{A}$ and $\mathcal{B}$, either find a homomorphism $h : \mathcal{A} \to \mathcal{B}$ if it exists, or else prove that no such homomorphism exists.

A bijective map $f : \mathcal{A} \to \mathcal{B}$ is called an *isomorphism* just in case both f and its inverse f^{-1} are homomorphisms.

4 Structure and category theory

In the last ten years or so, *category theory* has gained much purchase among those working on the foundations of physics, cf. in particular the work by John Baez, Jeremy Butterfield, and Chris Isham. Many believe that "categories" represent the basic "structures" in mathematics. In this vein, one can think of categories as an instantiation of structures as defined above, defined in such general terms that a lot of mathematics can be cast into its form. Let's do a little bit of basic category theory to appreciate this point. For this, I heavily rely on Baez (cf. References).

A *category* is an ordered pair $\langle \mathcal{O}, \mathcal{M} \rangle$ of a set $\mathcal{O}$ of *objects* and a set $\mathcal{M}$ of *morphisms* such that every morphism has a source and a target object in $\mathcal{O}$. For instance, for an $f \in \mathcal{M}$ with $X \in \mathcal{O}$ as its source and $Y \in \mathcal{O}$ as its target, one writes $f : X \to Y$. One usually denotes the set of morphisms from X to Y by $Hom(X, Y)$.

A category must, furthermore, satisfy the following axioms:

1. For any morphisms $g : X \to Y$ and $f : Y \to Z$, there exists a morphism $fg : X \to Z$ called the *composite* of f and g.

2. This composition is *associative*: $(fg)h = f(gh)$.

3. For every object $X \in \mathcal{O}$, there exists a morphism 1_X from X onto itself called the *identity* on X.

4. Composition satisfies the *left* and *right unit laws*: for any morphism $f : X \to Y$, we have $1_Y f = f = f 1_X$.

In this more general setting, an *isomorphism* is a morphism $f : X \to Y$ for which there exists an "inverse" morphism $f^{-1} : Y \to X$ such that $f^{-1}f = 1_X$ as well as $ff^{-1} = 1_Y$.

The standard example of a category is **Set**, whose objects are sets and whose morphisms are functions between the sets. The usual composition of functions satisfies the above stated axioms. Baez lists the following additional examples of categories:

- **Vect**: vector spaces as objects, linear maps as morphisms

- **Group**: groups as objects, homomorphisms as morphisms

- **Top**: topological spaces as objects, continuous functions as morphisms

- **Diff**: smooth manifolds as objects, smooth maps as morphisms

- **Ring**: rings as objects, ring homomorphisms as morphisms

5   Structure and model theory

According to a standard textbook in the field, "[m]odel theory is a branch of mathematical logic where we study mathematical structures by considering the first-order sentences true in those structures and the sets definable by first-order formulas." (Marker 2002, 1) We are not going to do full-blown model theory, but want to get a few definitions that will allow us to see how it conceives of structures. Intuitively, in model theory a structure is a set equipped with some collection of functions, relations, and elements. Thus, the intuitve characterization of structures is the one we started out with above in Section 3. But let's build this up a bit more carefully, following Marker (2002).

A *language* $\mathcal{L}$ is given by specifying the following: (i) a set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$; (ii) a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$; (iii) a set of constant symbols $\mathcal{C}$. The number n_f indicates that f is a function of n_f variables, and n_R tells us that R is an n_R-ary relation. Note that any or all of these sets may be empty. Here are some examples of languages:

- The language of *rings* is $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$, where $+, -$, and $\cdot$ are binary function symbols and 0 and 1 are constants.

- The language of *ordered rings* is $\mathcal{L}_{\mathrm{or}} = \mathcal{L}_r \cup \{<\}$, where $<$ is a binary relation symbol.

- The language of *graphs* is $\mathcal{L}_G = \{R\}$, where R is a binary relation symbol.

What, then, is a structure for which $\mathcal{L}$ is the appropriate language? An *$\mathcal{L}$-structure* $\mathcal{M}$ is given by the following:

(i) a non-empty set M called the *universe*, *domain*, or *underlying set* of $\mathcal{M}$;

(ii) a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$;

(iii) a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$;

(iv) an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$.

As in Section 3, $f^{\mathcal{M}}, R^{\mathcal{M}}$, and $c^{\mathcal{M}}$ are the *interpretations* of the symbols f, R, and c. Typically, the structure is written as $\mathcal{M} = \langle M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C} \rangle$.

As an example, let's look at groups. Consider the language $\mathcal{L}_g = \{\cdot, e\}$ with the binary function symbol $\cdot$ and the constant symbol e. An example of an $\mathcal{L}_g$-structure, then, is $\mathcal{G} = \langle G, \cdot^{\mathcal{G}}, e^{\mathcal{G}} \rangle$ of a set G equipped with a binary function $\cdot^{\mathcal{G}}$ and an element $e^{\mathcal{G}}$. For instance, $\mathcal{G} = \langle \mathbb{R}, \cdot, 1 \rangle$ is an $\mathcal{L}_g$-structure with the usual multiplication $\cdot^{\mathcal{G}} = \cdot$ and $e^{\mathcal{G}} = 1$. Also, graphs, to be discussed in the next section, are structures in the model-theoretic sense.

6   Structure and graph theory

Graph theory offers an alternative mathematical approach to represent structures. According to it, graphs are mathematical structures consisting of only two types of basic objects, "vertices" (or "nodes" or "points") and "edges" (or "links" or "lines") between vertices. Graphs can be partitioned into *directed* and *undirected* graphs, depending on whether the relation represented by the edges between vertices is symmetric or not.[2] Graphs can be defined set-theoretically (and

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[2]A *partition* of a set A is the set $\mathcal{A} = \{A_1, ..., A_k\}$ of disjoint subsets of A such that the union $\cup \mathcal{A}$ of all sets $A_i \in \mathcal{A}$ is A and $A_i \neq \emptyset$ for every i.
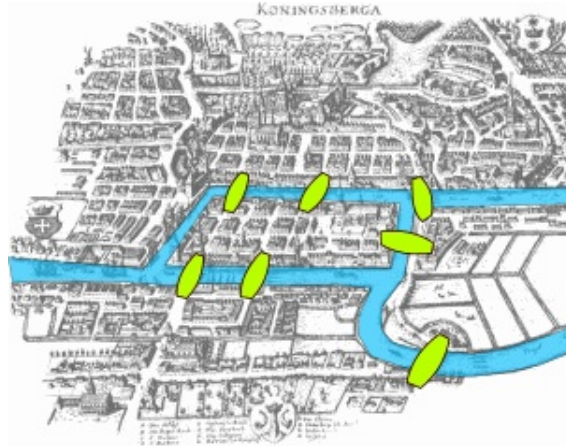
Figure 1: The *Seven Bridges of Königsberg*: Can you find a path through the city that crosses each bridge exactly once? This problem was solved negatively by Leonhard Euler in 1735 whose analysis of the problem introduced graph theory.

thus in terms of our discussion above) as ordered pairs of a set of vertices and a set of edges where the latter are interpreted as ordered pairs of vertices in directed graphs and as unordered pairs in undirected graphs. Leitgeb and Ladyman (2008) claim that this set-theoretic characterization of graphs does not do justice to the structural content of graph theory and the actual practice of its practitioners. Instead, they propose to start out from *unlabelled graphs*, where different vertices are taken to be indistinguishable if considered in isolation. In fact, this means that only *abstract graphs* are considered, i.e. equivalence classes of labelled graphs under isomorphisms. Labelled graphs are unlabelled graphs with the additional stipulation that the vertices are distinguished by, well, you guessed it, labels. But let's look at the basics of what a graph is more rigorously (cf. Diestel).

Let us denote by $[A]^k$ the set of all subsets of A with k elements. A *graph*, then, is an ordered pair $G = \langle V, E \rangle$ of sets such that $E \subseteq [V]^2$, i.e. the elements of E are 2-element subsets of V, where it is tacitly assumed that $V \cap E = \emptyset$. The elements of V are called *vertices* of the graph G and those of E are its *edges*. Graphs can be drawn by representing vertices by dots and edges as lines connecting them if the corresponding two vertices form an edge. How the dots are placed on a sheet of paper and how the lines are drawn is irrelevant, all that matters is to represent the information of which pairs of vertices are connected by an edge and which ones are not. Thus, the two drawings in Fig. 3 represent the same graph. A graph with vertex set V is said to be a graph *on* V. The *order* $|G|$ of a graph G is the number of its vertices. The number of its edges is
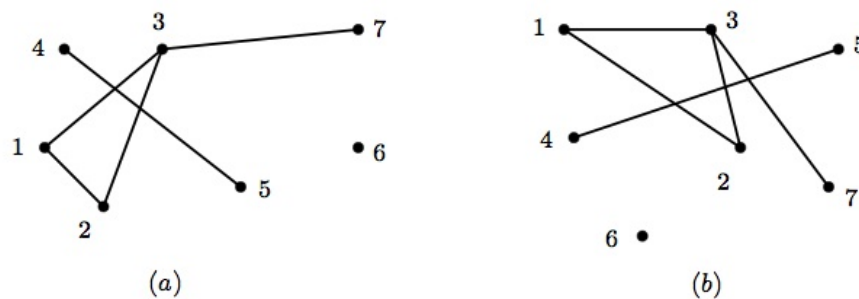

Figure 2: The graph on $V = \{1, ..., 7\}$ with edge set $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 7\}, \{4, 5\}\}$.

denoted by $\|G\|$.

We write an edge $\{x, y\}$ as xy (or yx). Two graphs $G = \langle V, E \rangle$ and $G' = \langle V', E' \rangle$ are *isomorphic*, denoted $G \simeq G'$, just in case there exists a bijection $\phi : V \to V'$ with $xy \in E \Leftrightarrow \phi(x)\phi(y) \in E'$ for all $x, y \in V$. A map ϕ like this is called an *isomorphism*; in case $G = G'$, it is called an *automorphism*. An *abstract graph* is an equivalence class of isomorphic graphs. Thus, a structuralist is naturally interested in abstract graphs.

A *graph property* is a class of graphs closed under isomorphism. For instance, "containing a triangle" is a graph property since if a graph G contains three pairwise adjacent vertices, then so does every graph isomorphic to it. A *graph invariant* is a map taking graphs as arguments that assigns equal values to isomorphic graphs. For instance, $|G|$ and $\|G\|$ are two simple graph invariants.
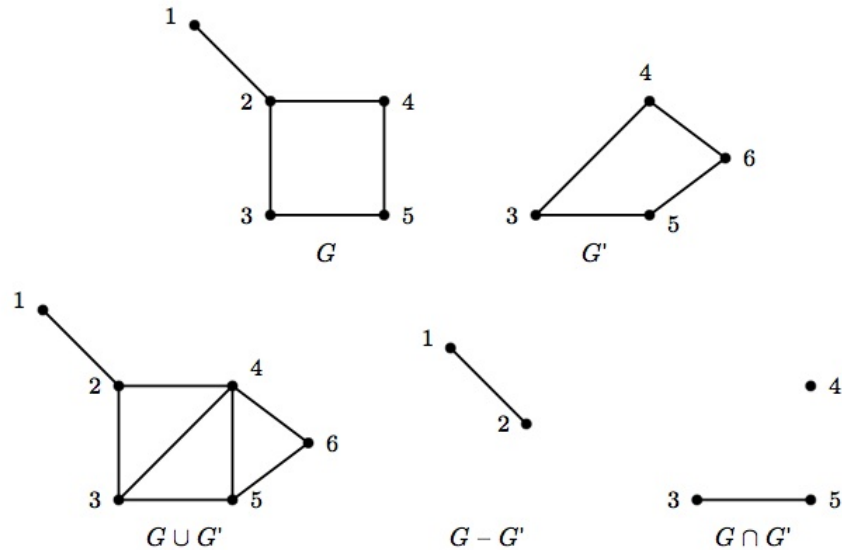

Figure 3: Union, difference, and intersection (adapted from Diestel (2006, 3)).

The *union* of two graphs is defined as $G \cup G' := \langle V \cup V', E \cup E' \rangle$ and their *intersection* as $G \cap G' := \langle V \cap V', E \cap E' \rangle$. Two graphs are *disjoint* if $G \cap G' = \emptyset$. In case $V' \subseteq V$ and $E' \subseteq E$, G' is a *subgraph* of G (and G a *supergraph* of G'), denoted $G' \subseteq G$. If $G' \subseteq G$ and for all $x, y \in V'$, G' contains all the edges $xy \in E$, then we say that G' is an *induced subgraph* of G and that V' *induces* or *spans* G' in G, denoted $G' =: G[V']$. Informally speaking, the set-theoretic *difference* of two graphs, $G - G'$, is obtained from G by deleting all the vertices in $V \cap V'$ and their incident edges. The *complement* $\bar{G}$ of G is defined as $\langle V, [V]^2 \setminus E \rangle$.

For purposes that may or may not become apparent in the course of this seminar, I am also interested in finding an *inversion* operation * that relates two graphs G and G^* just in case the elements of E are taken to stand in a bijective relation to the elements of V^* and E^* is constructed such that it contains elements that bijectively correspond to elements in V. For an appropriate inversion or duality relation like this, under which general conditions are a graph and its inversion or dual isomomorphic to one another?

7 A ridiculously brief history of structuralism

Structuralism, just like graph theory, started with a Swiss. It was first formulated by Ferdinand de Saussure as a thesis in linguistics (graph theory, as we have seen above, was initiated by

Euler). The main idea is, of course, to analyze a field as a complex system of interrelated parts. The idea has spread to many other field, such as sociology, anthropology, psychology, philosophy, mathematics, physics,[3] psychoanalysis, literary theory, and architecture, just to name a few, and has led to many different expressions of it in all these fields. Even within philosophy, structuralism can refer to different ideas and positions.[4] Part of our job in this seminar is to at least pursue some of its strands in philosophy and the foundations of mathematics and physics.

In mathematics, the concept of a structure (but not yet structuralism) gained currency with Felix Klein's "Erlangen programme" of 1872, which was an attempt at a "synthesis of geometry as the study of the properties of a space that are invariant under a given group of transformations."[5] Starting in the 1930s, structuralism in mathematics reached new levels of rigour at the hands of the French group of mathematicians who called themselves *Nicolas Bourbaki*. Chapter 4 ("Structures") of their influential book *Théorie des ensembles* (*Set Theory*) explicates a very general notion of "structure" and of "isomorphism".

References

[1] Baez, John. "Categories, quantization, and much more". Available at http://math.ucr.edu/home/baez/categories.html.

[2] Bourbaki, Nicolas. *Éléments de mathématiques: Théorie des ensembles*. Springer: Berlin, 2006.

[3] Diestel Reinhard, *Graph Theory*, Springer, Berlin, ³2006.

[4] Hodges, Wilfrid, *Model Theory*, Cambridge University Press, Cambridge, 1993.

[5] Landry, Elaine, "Shared structure need not be shared set-structure", *Synthese* **158** (2007): 1-17.

[6] Leitgeb, Hannes and Ladyman, James, "Criteria of identity and structuralist ontology", *Philosophia Mathematica* **16** (2008): 388-396.

[7] Marker, David. *Model Theory: An Introduction*. Springer: New York, 2002.

[8] Muller, Fred, "The characterisation of structure: Definition versus axiomatisation", to appear in the *Proceedings of an ESF Conference*, Vienna, December 2008.

[9] Roberts, Bryan, "Group structural realism", forthcoming in *British Journal for the Philosophy of Science*.

Additional resources used:

- Planet math entries (http://planetmath.org/): "Structure", "Signature"

- Wikipedia entries: "Isomorphism", "Mathematical structure", "Seven Bridges of Königsberg"

- Wolfram MathWorld entries (http://mathworld.wolfram.com/): "Isomorphism"

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[3] Cf. e.g. http://plato.stanford.edu/entries/physics-structuralism/.

[4] Cf. e.g. the Wikipedia entry on structuralism in philosophy of science, which seems to conceive of structuralism differently from the mainstream literature around structural realism and its structuralist cousins in the foundations of physics.

[5] http://www-history.mcs.st-andrews.ac.uk/Biographies/Klein.html