Introduction: What is philosophy of mathematics?

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What is philosophy of mathematics?

(near quotes from Shapiro's preface)

1. Matters of metaphysics:
   - What is mathematics about? Does it have a subject-matter? What are numbers, sets, points, lines, functions, etc.?
2. Matters of semantics:
   - What do mathematical statements mean? What is the nature of mathematical truth?
3. Matters of epistemology:
   - How is mathematics known? What is its methodology? Is observation involved, or is it a purely mental exercise? What is a proof, and are they absolutely certain? Are there unknowable mathematical truths?

Stewart Shapiro, *Thinking About Mathematics*

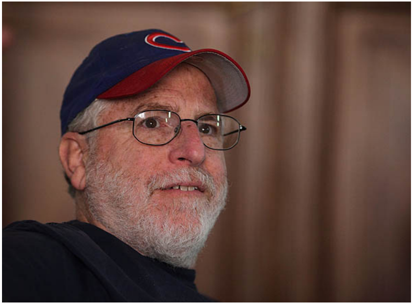


Photo: A. Walanus

Philosophy of mathematics belongs to a genre that includes philosophy of physics, philosophy of biology, philosophy of psychology, philosophy of language, philosophy of logic, and even philosophy of philosophy.

The theme is to deal with philosophical questions that concern an academic discipline, issues about the metaphysics, epistemology, semantics, logic, and methodology of the discipline. Typically, philosophy of X is pursued by those who care about X, and want to illuminate its place in the overall intellectual enterprise. Ideally, someone who practises X should gain something by adopting a philosophy of X: an appreciation of her discipline, and orientation toward it, and a vision of its role in understanding the world. The philosopher of mathematics needs to say something about mathematics itself, something about the human mathematician, and something about the world where mathematics gets applied. A tall order. (p. viif)

Our topics

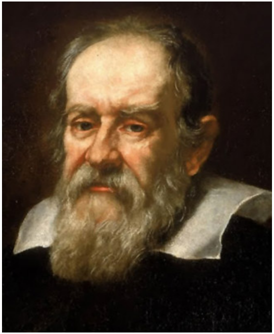
1. Introduction: What is philosophy of mathematics?
2. From Plato to Mill
3. Set theory
4. Logicism
5. Formalism

6. Intuitionism
7. Do numbers exist?
8. Structuralism

OR

6. Transfinite mathematics
7. Löwenheim-Skolem thm
8. Gödel's theorem
9. Intuitionism
10. Structuralism

Galileo Galilei (1564-1642)

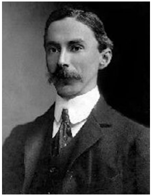


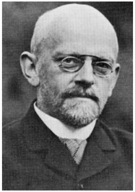
*Philosophy is written in this grand book—I mean the universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth. (*Il Saggiatore *(The Assayer), 1623)*

Mathematician-philosophers










Rationalism and Empiricism

- Rationalism: attempt to extend methodology of mathematics to other enquiries
- Empiricism: if all knowledge must ultimately be grounded in experience, then how can we have mathematical knowledge when mathematics is based on proof, not observation?

Connections between mathematics and philosophy

- historical connection
- mathematical techniques and tools (model-theoretic semantics, possible-worlds analyses of modal and epistemic discourse, extensional understanding of properties and relations, λ-calculus in philosophy of language)
- mathematics is important case study for philosophers (on questions of reference, matters of normativity)
- epistemology: mathematics is primary tool in our best effort to understand world
  - $\Rightarrow$ Shapiro: "This suggests that philosophy of mathematics is a branch of epistemology, and that mathematics is an important case for general epistemology and metaphysics." (6)

Relationship between math and philosophy of math

‘Philosophy-first’ or ‘philosophy-last-if-at-all’?

To what extent can we expect philosophy to determine or even suggest the proper practice of mathematics? Conversely, to what extent can we expect the autonomous practice of mathematics to determine the correct philosophy of mathematics? (Shapiro, 7)

The philosophy-first principle

Philosophical considerations concerning what mathematics is about precedes the practice of mathematics “in some deep metaphysical sense.” (10)

$\Rightarrow$ philosophical questions, e.g. concerning metaphysics and ontology, determine proper practice of mathematics... let’s look at some examples

Philosophical challenges to mathematics

(1) Platonism

- Platonism maintains that subject matter is ideal, eternal, unchanging realm of abstract forms
- Euclid: between any two points one can draw a straight line

$\Rightarrow$ Plato: impossible! Mathematicians don't know what they talk about and do it incorrectly.

- Hilbert: between any two points there is a straight line
- but: long-standing problems of math don't concern existence (trisecting an angle, squaring a circle, doubling a cube...)

Philosophical challenges to mathematics

(2) Intuitionism

Mathematics rigorously treated from [the] point of view [of] deducing theorems exclusively by means of introspective construction, is called intuitionistic mathematics... [I]t deviates from classical mathematics... because classical mathematics believes in the existence of unknown truths. (L E J Brouwer 1948)

Arend Heyting (doctoral student of Brouwer):

> *Brouwer's programme... consisted in the investigation of mental mathematical construction as such... In the study of mental mathematical constructions, 'to exist' must be synonymous with 'to be constructed'... In fact, mathematics, from the intuitionistic point of view, is a study of certain functions of the human mind. (1956)*

- ⇒ deny the law of excluded middle ('For any proposition *p*, either *p* is true or it is not') and principles based on it because they are "symptomatic of faith in the transcendental existence of mathematical objects and/or the transcendental truth of mathematical statements" (Shapiro, 9)
- ⇒ revisionary new logic: intuitionistic logic (instead of 'classical logic')

Classical vs. intuitionistic logic

Consider the following propositions:

(p) 'Not all elements of a set have a certain property P', formally $\neg\forall x Px$.

(q) 'There is an element which lacks property P', formally $\exists x \neg Px$.

Intuitionistic logic:

- content of p is that it is refutable that one can find a mental construction showing that P holds of all elements
- content of q is that one can construct an element of the set and show that it does not exemplify P
- Clearly, $q \supset p$, but not the converse because it's "possible to show that a property cannot hold universally without constructing a number for which it fails." (9)

Classical logic:

- content of p and q realistically given

$\Rightarrow$ $p \Leftrightarrow q$

Philosophical challenges to mathematics

(3) Predicative mathematics

Definition (Impredicative definitions)

"A definition of a mathematical entity is impredicative if it refers to a collection that contains the defined entity." (9)

- Examples: 'least upper bound'
- Henri Poincaré: impredicative definitions illegitimate because they presuppose existence of what is to be defined $\Rightarrow$ viciously circular
- Kurt Gödel: defence of impredicative definitions based on realism in mathematics (analogous to 'village idiot')

A problem for the philosophy-first principle

- ... it does not reflect the history of mathematics and its actual practice
- Even in last example, reason why mathematicians use impredicative definitions is not because they are all realists (even though they may be), but because they are essential to the practice of mathematics—as is the law of the excluded middle.
- (Gödel argues that realism conforms well with practice, but is not based on first principles)
- mathematics is what mathematicians do, and the job of philosophy of math is to offer a coherent account of it
- Michael Dummett: argument from philosophy of language to the conclusion that classical logic must be replaced by intuitionistic

$\Rightarrow$ Which is more secure: maths as practiced or Dummett's philosophy of language?

An opposite extreme: ‘philosophy-last-if-at-all’

The philosophy-last-if-at-all principle

Philosophy is irrelevant to mathematics and its practice. It has nothing to contribute to the mathematical enterprise, except perhaps to give a coherent and extremely fallible account of mathematics as practiced up to now.

- Even though many (most?) practising mathematicians are not interested in philosophy, this principle also doesn’t reflect the history of mathematics.
- As we will see in the course of this course, philosophy contributed to the clarification of some important problems in the foundations of mathematics, has lead to progress in some instances, or at least set the direction of mathematical research.

A happy middle ground

Shapiro:

> *No practice is sacrosanct... Philosophy and mathematics are intimately interrelated, with neither one dominating the other. On this view, the correct way to do mathematics is not a direct consequence of the true philosophy, nor is the correct philosophy of mathematics an immediate consequence of mathematics as practised... As I see it, the primary purpose of the philosophy of mathematics is to interpret mathematics, and thereby illuminate the place of mathematics in the overall intellectual enterprise. (14ff)*

Mathematics and naturalism

For Willard V O Quine, naturalism is...

the abandonment of the goal of first philosophy... [and] the recognition that it is within science itself... that reality is to be identified and described. (1981, 72)

$\Rightarrow$ rejection of philosophy-first principle

- However, Quine himself accepts mathematics only insofar as it is needed in science; he maintains that "if a part of mathematics does not play an inferential role (however indirect) in the parts of the scientific web that bear on sensory perception, then that part should be jettisoned" (Shapiro, 18)
- But: goals of scientific enterprise well served by mathematicians pursuing whatever they want with their methodology

Penelope Maddy's naturalism

- rejection of radical Quinean holism, takes 'seams' seriously
- her naturalism "prescribes a deferential attitude towards mathematicians" (18)
- proposes similar attitude towards maths as did Quine towards science
- But with Quine, she maintains that there is no need for first philosophy.

Necessity and a priori knowledge

- mathematical truths appear to be necessary, quite unlike true propositions about the physical world
- This is underwritten by the fact that they can only be established by proof, eliminating rational doubt.
- Q: what is necessity?
- mathematical statements also seem to be known a priori, i.e. known 'prior to, or independent of, experience', as opposed to a posteriori (= not known a priori)

$\Rightarrow$ "It is thus incumbent on any complete philosophy of mathematics to account for the at-least apparent necessity and a priority of mathematics." (23)

A tension in the traditional view

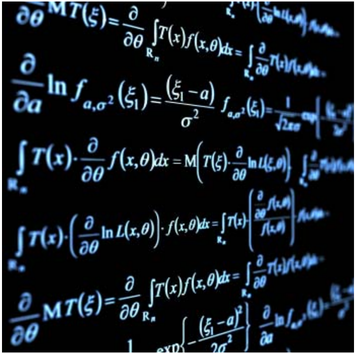
- The traditional view as sketched by Shapiro thus suffers from a serious tension: it maintains that mathematical knowledge is, on the one hand, necessary and a priori, yet, on the other hand, it "has something to do with our physical world." (23)
- valiant effort by Kant to ease this tension: conceptual unification and integration by active mind using 'precepts' (space, time) and following 'categories of understanding' (cause, substance) operating on manifold of sense perceptions

$\Rightarrow$ mental 'structuring' of sense perceptions a priori given; mathematics part of what describes these structures

- general problem of Kantian approach: invokes intuition and inherits all difficulties about it

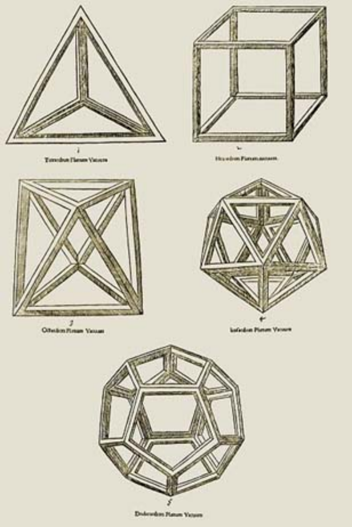
$\Rightarrow$ need for non-Kantian approach

Global matters: objects and objectivity



- metaphysical and epistemological issues
- metaphysical: What is the subject matter of mathematics? What are its objects?
- epistemological: What are its methods, and are they reliable? To what extent are its principles objective independent of mind, language, social structure of mathematicians? Is every mathematical truth knowable?

(1) Objects



Q: Do mathematical objects (sets, numbers, lines, functions, circles, categories, etc.) exist?

Definition (Realism in ontology)

At least some mathematical objects exist objectively and independently of the human mind.

Realism in ontology is also often called Platonism. Opposed to realism are two views: idealism and nominalism.

Realism's opponents

Definition (Idealism)

Idealism agrees that mathematical objects exist, but insists that they depend on the human mind in the sense that if there were no minds, there wouldn't be mathematical objects.

Definition (Nominalism)

Nominalism (concerning mathematical objects) denies the objective existence of mathematical objects.

Nominalism

Nominalism comes in at least two varieties:

Definition (Linguistic nominalism)

Mathematical objects are mere linguistic entities, i.e., they coincide with their names. For instance, numbers are nothing but the numerals that we ordinarily think denote them.

Definition ((Nihilistic) nominalism)

This version of nominalism simply denies the existence of mathematical objects and does not care whether they can be linguistically constructed.

A brief assessment

- realism in ontology can easily account for the necessity of mathematics (how?), but faces a real challenge regarding a priority (unless one assumes, à la Gödel, a quasi-mystical connection between the Platonic objects and our minds): how can material objects like humans living in a physical world have knowledge, indeed a priori knowledge, of abstract, eternal, acausal objects like sets and numbers? $\Rightarrow$ deep epistemic mystery for the realist
- idealism may well be able to straightforwardly account for both the necessity and the a priority; but the deeper problem is to reconcile the finitude of our minds with knowledge of infinite mathematical structures
- nominalists must either reconstruct mathematical proposition as involving no reference to objects (and then account for the apparent necessity and a priority), or else hold that mathematical propositions are systematically false (and hence not necessary) and therefore cannot be known (and hence are not a priori)

(2) Truth: Is mathematical discourse objective?

Definition (Realism in truth-value)

Mathematical statements have objective truth-values, independent of minds, languages, conventions, social structures of mathematicians.

Definition (Antirealism in truth-value)

"[I]f mathematical statements have truth-values at all, these truth-values are dependent upon the mathematician." (29)

- one version of antirealism is idealism in truth-value, according to which truth-values depend on mind; in some sense, we make mathematical statements true or false (but don't necessarily decide them to be either way)
- for the realist, some mathematical truths may be unknowable because forever beyond our epistemic grasp; for the antirealist may argue that all mathematical truths are knowable

Antirealism in truth-value

- antirealists may reject bivalence and instead replace classical logic with intuitionistic logic (because mind may not determine a truth-value for every unambiguous mathematical statement)
- more radically, she may deny that mathematical statements have truth-values altogether; but then what is good mathematics supposed to be if not true mathematics?

Paul Benacerraf, 'Mathematical truth' (1973)

Paul Benacerraf, 'Mathematical truth', *Journal of Philosophy* **70** (1973): 661-679. Reprinted in Benacerraf and Hilary Putnam, *Philosophy of Mathematics*, Cambridge University Press (^{2}1983).

Dilemma (Benacerraf 1973)

EITHER it is very hard to understand how we can come to know any mathematical truths OR we must give up a semantic continuity of mathematical with everyday and scientific discourse (OR we must give up our usual semantics of ordinary and scientific languages)

Semantic desideratum

"[M]athematical statements should be understood in the same way as ordinary statements, or at least respectable scientific statements. That is, we should try for a uniform semantics that covers ordinary/scientific language as well as mathematical language." (Shapiro, 31)

The first horn: double realism

- appreciate the natural alliance of realism in ontology with realism in truth-value: mathematical statements deal with objective features of the world, and the terms of its language denote (e.g. numerals denote numbers); while logically independent, that mathematical objects really exist in their own right is suggested by the objective truth of mathematical assertions
- realism in truth-value leads to a straightforward satisfaction of the semantic desideratum: if a realism in truth-value holds for scientific (or ordinary) discourse, and our realism guarantees that this is also the case for mathematics, then the desideratum is satisfied
- But if mathematical objects objectively exist, and they are abstract entities without causal nexus to our material world, how can we come to know anything about them?

$\Rightarrow$ 'double realism' nicely satisfies the semantic continuity criterion, but at the expense of a deep epistemological puzzle

The second horn: double antirealism

- idealism in ontology makes it natural to accept an idealism in truth-value: mathematical truth about mental mathematical objects depends on the mind
- similarly for other forms of antirealism: one's ontological position is naturally associated with one's analysis of mathematical truth (e.g. Hartry Fields 1980: nominalism about mathematical objects, combined with an assertion that mathematical statements have vacuous truth-values

$\Rightarrow$ no (or little) epistemological puzzle: either there is a sense in which there is nothing to 'know', or it's all mental/ideal, in which case there's a straightforward connection in the mind

- However, if one wants to maintain the semantic continuity desideratum, one is committed to denying realism about both ordinary and scientific discourse—an option many found unattractive.

$\Rightarrow$ One gets a straightforward epistemic account, but at the expense of either rejecting the semantic desideratum or accepting antirealism about ordinary and scientific discourse

Where does this leave us?

- The natural connection between attitudes toward mathematical objects and objectivity are not logical!
- Shapiro: “Each of the four possible positions is articulated and defended by established and influential philosophers of mathematics.” (33)
- E.g.: attempt to account for objectivity of mathematical discourse without postulating objective existence of mathematical objects (Charles Chihara and Geoffrey Hellman); this is a combination of antirealism in ontology with realism in truth-value
- Neil Tennant: realism in ontology and antirealism in truth-value
- double realists: Kurt Gödel, Penelope Maddy, Michael Resnik, Stewart Shapiro
- double antirealists: Hartry Field, Michael Dummett, L E J Brouwer, Arend Heyting

The mathematical and the physical

A central question

How can mathematics explain any physical fact or regularity at all? Given that this is constantly done in science, we need an account of how mathematics relates to the physical world.

There are really a number of distinct questions (Steiner 1995):

- semantic problem: “find an interpretation of the language that covers ‘pure’ and ‘mixed’ contexts, so that proof within mathematics can be employed directly in scientific contexts.” (36)
- metaphysical problem (for realism and idealism alike): relation between mathematical objects and physical world
- problem of applicability: explain “[t]he unreasonable effectiveness of mathematics in the natural sciences”, as Eugene Wigner (1960) pointed out

On the problem of applicability/explanation

Three levels of questions:

1. How can a particular physical event be explained by a particular mathematical fact? (Remember the story of Shapiro's friend)
2. How can an entire mathematical theory, or class of objects be relevant in explanations of (classes of) physical events or theories about them?
3. Why is mathematics quite generally so essential to science? (Note that the Quine-Putnam indispensability argument (§8.2) doesn't explain why mathematics is indispensable to science)

Local matters: theorem, theories, and concepts

(A) One set of important issues in the philosophy of mathematics pertain to particular mathematical or logical results/theorems; e.g.,

1. Skolem's paradox, trading on the Löwenheim-Skolem theorems
2. issues in set theory, such as Georg Cantor's continuum hypothesis
3. Kurt Gödel's incompleteness theorem

(B) Another set of issues surround articulations and interpretations of mathematical theories and concepts.

Warning

The next three slides contain material which goes beyond what you are now expected to master and, for the first and the third slide, possibly (depending on chosen path) beyond what you are expected to know by the end of this class.

(1) Skolem's paradox

- suppose T is a formal mathematical theory and M is a mathematical structure (e.g., $\mathbb{N}$, or $\mathbb{R}$)
- If T is true of M, then we say that M is a model of T.
- The ('compactness theorem' and the) 'Löwenheim-Skolem theorems' entail that if a 'first-order theory' has an infinite model, then for any infinite cardinality λ, the theory has a model of exactly size λ.
- This has counterintuitive consequences known as 'Skolem's paradox':
    - $\exists$ models of first-order real analysis and first-order set theory (e.g., Zermelo-Fraenkel set theory) with the cardinality of $\mathbb{N}$ (even though Georg Cantor showed that in set theory, the cardinality of $\mathbb{R}$ and of sets is strictly larger than that of $\mathbb{N}$)
    - 'first-order arithmetic' (theory of the natural numbers) has models which are larger than $\mathbb{N}$, e.g. with the cardinality of $\mathbb{R}$

$\Rightarrow$ suggests that first-order theories such as arithmetic and real analysis do not have a fixed subject-matter, and their terms may thus not have a fixed reference (even though what the philosophical consequences are is far from settled)

(2) Set theory and Cantor's continuum hypothesis

- Zermelo-Fraenkel set theory with the axiom of choice, or 'ZFC set theory' for short, is very expressive and powerful theory, yet there are central mathematical questions which are undecidable in it.
- E.g., it can be shown that neither Cantor's continuum hypothesis nor its negation can be proved in ZFC.

Cantor's continuum hypothesis

There are no sets which are strictly larger than the set of natural numbers and strictly smaller than the set of real numbers.

Q : Does this militate against realism in truth-value?

(3) Gödel's incompleteness theorem

An informal statement of Gödel's incompleteness theorem

"Let T be an axiomatization of arithmetic. Assume that T is effective, in the sense that there is a mechanical procedure to determine whether a sequence of sentences in the language of T is a correct derivation in T. Roughly, the incompleteness theorem entails that if T is sufficiently rich, then there is a sentence Φ in the language of T such that neither Φ nor its negation is derivable in T. In other words, T does not decide Φ." (43)

- Again, this may be taken to support antirealism in truth-value since it seems as if not all arithmetic statements have determinate truth-values.
- Shapiro: this presupposes that only route to truth is via proof in a deductive system