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Formalism

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Formalism, a first approach

The various philosophies that go by the name of 'formalism' pursue a claim that the essence of mathematics is the manipulation of characters. A list of the characters and allowed rules all but exhausts what there is to say about a given branch of mathematics. According to the formalist, then, mathematics is not, or need not be, about anything, or anything beyond typographical characters and rules for manipulating them. (Shapiro, 140)

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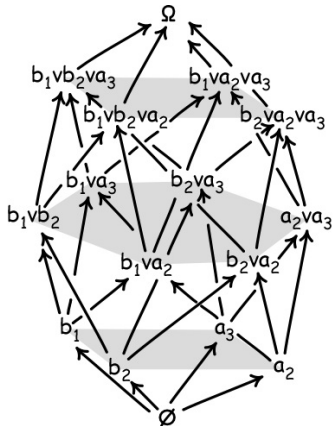
Formalism: five characteristics

Michael Detlefsen, 'Formalism', in S. Shapiro (ed.), *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford University Press, 2005): 236-317.
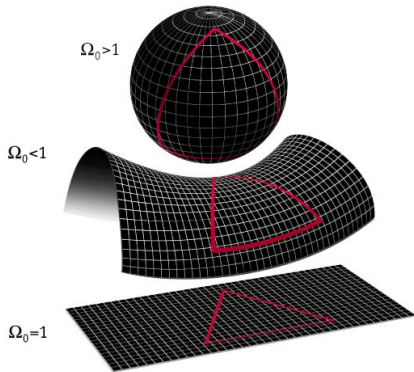
Formalism is characterized by

1. a rejection of the previously accepted priority of the 'science of magnitude' (geometry) over the 'science of multitude' (arithmetic) and a reversal of this ordering;

2. a rejection of the classical Aristotelian "genetic conception of proof by denying that the only proper knowledge of a thing comes through knowledge of its causes" (237);

3. "a conception of rigor that emphasized abstraction from rather than immersion in intuition and meaning" (ibid.), i.e., a retreat from intuition;

4. an "advocacy of a nonrepresentational role for language in mathematical reasoning" (ibid.);

5. the "creativist component" of asserting the freedom to create whatever instruments of reasoning the mathematician deems conducive to her epistemic ends.

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Historical enablers of formalism

(i) emergence of algebraic methods

(ii) erosion of authority of Euclidean geometry

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Term formalism
Game formalism
Between Permanence and Solvability

(1) Term formalism

Definition (Term formalism)

"Term formalism is the view that mathematics is about characters or symbols—the sysmtes of numerals and other linguistic forms. That is, the term formalist identifies the entities of mathematics with their names. The complex number $8 + 2i$ is just the symbol '$8 + 2i$'." (Shapiro, 142)

⇒ maths has subject matter, propositions are true or false

- What is maths about? Numbers, sets, functions, etc.

- But what are these numbers etc? Merely linguistic characters.

- How do we know mathematical propositions? Mathematical knowledge is of how the characters are related to one another and how they are manipulated.

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Frege's attack on term formalism

- Frege: symbol '$5 + 7$' is not identical to symbol '$6 + 6$', and term formalist can't claim that they denote the same number ($\neg\exists$ extralinguistic entities denoted by numerals)

- If terms denote just the characters themselves, '$=$' can't be interpreted as identity.

- Frege (on behalf of term formalist): symbol '$5 + 7$' can be substituted anywhere, *salva veritate*, for '$6 + 6$'

- But how does this story work for real numbers, where most real numbers don't have names? (decimal expansions of reals are infinitary objects, not linguistic symbols)

- Furthermore: can at best deal with calculation, but not with propositions (How is the prime number theorem about symbols?)

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(2) Game formalism

Definition (Game formalism)

Game formalism asserts that mathematical symbols are meaningless or, more moderately, only have meanings which are irrelevant to mathematical practice. Thus, the characters of a mathematical theory lack a mathematical interpretation, i.e., mathematics has no subject-matter. "Mathematical formulas and sentences do not express true or false propositions about any subject-matter... The 'content' of mathematics is exhausted by the rules for operating with its language." (144) In other words, "mathematics is about its terminology." (145)

- So what is maths about? Nothing.
- What are mathematical objects such as numbers, etc? They might as well not exist.
- How do we know mathematical propositions? Mathematical knowledge is of rules of the game or that such and such a move accords with the rules.

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Another Fregean criticism

If the meaning of mathematical terms is extraneous to mathematics, how come our mathematical 'games' are so useful in the empirical sciences?

> *[A]n arithmetic without thought as its content will also be without possibility of application. Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought. If it did so and every chess move conforming to the rules corresponded to a transition from one thought to another, applications of chess would also be conceivable. Why can arithmetical equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? [I]t is applicability alone which elevates arithmetic from a game to the rank of science. (Frege, **Gg** 2/1903, §91, my emphasis)*

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Between Permanence and Solvability

Principle (Permanence)

"Algebra should preserve (to the greatest extent possible) the arithmetic laws of the simplest quantities," the natural numbers. (Detlefsen, 278)

This principle immediately raises questions:

- To what extent?

- What counts as a law?

But most importantly, we do want to extend our algebraic theories beyond the arithmetic of natural numbers, as Giuseppe Peano (1858-1932) reminds us:

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One... begins by premising certain propositions whose validity is in no way doubtful.

These are:

There does not exist a number (from the sequence 0, 1,...), which when added to 1 gives 0.

There does not exist a number (integral), which multiplied by 2 gives 1.

There does not exist a number (rational), whose square is 2.

There does not exist a number (real), whose square is $\sqrt{-1}$.

Then one says: in order to overcome such an inconvenience, we extend the concept of number, that is, we introduce, manufacture, create (as Dedekind says) a new entity, a new number, a sign, a sign-complex, etc., which we denote -1, or 1/2, or $\sqrt{2}$, or $\sqrt{-1}$, which satisfies the condition imposed. (1910, 224)

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The fundamental theorem of algebra

The complex numbers $\mathbb{C}$ form an algebraically closed set:

Theorem (Fundamental theorem of algebra)

"[E]very polynomial with complex coefficients has a complex solution." (Detlefsen, 279)

$\Rightarrow$ extension to ever more encompassing number systems reaches a natural end

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Hilbert's *ignorabimus* and the Axiom of Solvability

> *However unapproachable... problems may seem to us and however helpless we stand before them, we have, nevertheless, the firm conviction that their solution must follow by a finite number of purely logical processes. Is this axiom of solvability of every problem a peculiarity characteristic of mathematical thought alone, or is it possibly a general law inherent in the nature of the mind, that all questions which it asks must be answerable? ... The conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no* ignorabimus. *(Hilbert 1901, 444f)*

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How Permanence and Solvability play out

$\Rightarrow$ trade off between Solvability and Permanence, which acts as constraint on Solvability; and there are three complications:

1. What counts as a solution to a problem? As far as Hilbert is concerned, apparently quite a lot:
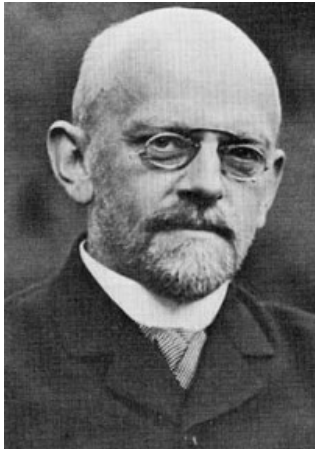
   > *... every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solutions and therewith the necessary failure of all attempts. (Hilbert 1901, 444)*

   $\Rightarrow$ How much difference is there between *ignorabimus* and *kein ignorabimus*?

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2 To what extent can we depart from the arithmetic laws of $\mathbb{N}$?

- can't be that we cannot at all, since $\mathbb{C}$ violates the law that "any number multiplied by itself is either 0 or positive" (283)—just think of $i \in \mathbb{C}$ [blackboard]
- ⇒ tension between Solvability and Permanence: without extension to $\mathbb{C}$, the fundamental law of algebra doesn't hold and numbers are not 'algebraically closed', but Permanence resists such an extension because of the failure of the above mentioned law for $\mathbb{C}$
- Hamilton's 'quaternions' and Graves's and Cayley's 'octonions' involve violations of basic arithmetic laws such as commutativity of multiplication (both) and associativity of multiplication (octonions)
- ⇒ less of a problem, since Solvability doesn't enjoin extending numbers beyond $\mathbb{C}$
- Still, what if quaternions or octonions turn out to have tremendously important applications?

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3 Given that some held that complex numbers don't offer genuine solutions to problems, does Solvability really sanction the extension to $\mathbb{C}$?

- Euler, Hamilton: $\sqrt{-1}$ is an absurdity
- Hamilton 1837: complex numbers as 'two-dimensional' quantities, i.e., pairs of two real-valued numbers
- $\Rightarrow$ multi-dimensional approach to quantity
- result known as 'Hankel's Theorem' "says that $\mathbb{C}$ is the most complete (or most nearly complete) of all the multidimensional elaborations of the number concept in the sense that it preserves the greatest portion of the standard laws of number." (286)
- $\Rightarrow$ "The complexes are the maximum required by the Axiom of Solvability. They are the maximum permitted by the Principle of Permanence." (287, my emphases)

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David Hilbert (1862-1943)

- PhD 1885 Königsberg

- initial appointment at Königsberg, moved to Göttingen in 1895

- invariant theory, axiomatization of geometry, functional analysis (Hilbert spaces), mathematical logic, metamathematics

- assistants: Hermann Weyl, John von Neumann, Paul Bernays

- students: Weyl, Zermelo, Hempel, Richard Courant, Haskell Curry

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Deductivism: Hilbert's *Grundlagen der Geometrie*

- Let's reconsider a variant of the moderate form of game formalism. Recall that Frege challenged its applicability. At a minimum, it seems as if for the application to succeed, the rules of the game are constrained in that they must constitute logical consequences (so that steps preserve truth).

⇒ doesn't matter how language is interpreted, since if axioms are true, so will be the theorems; this gives raise to

Definition (Deductivism)

"A deductivist accepts Frege's point that rules of inference must preserve truth, but she insists that the axioms of various mathematical theories be treated as if they were arbitrarily stipulated. The idea is that the practice of mathematics consists of determining logical consequences of otherwise uninterpreted axioms." (Shapiro, 149)

⇒ consonant with modern view that logic is 'topic-neutral'

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- So what is maths about? Nothing.
- What is mathematical knowledge? It is knowledge of valid inferences, it's logical knowledge.
- How is a mathematical theory applied? By furnishing the uninterpreted calculus with an interpretation that render the axioms true.
- So, deductivism rejects the Kantian thesis claiming a link of mathematics to intuitions, e.g., of space and time; thus, geometry should also be topic-neutral (qua mathematical theory).
- "The [formalist] programme executed in [Hilbert's] *Grundlagen der Geometrie* (1899) marked an end to an essential role for intuition in geometry." (151)
- Otto Blumenthal: in Berlin train station in 1891 Hilbert claimed that in proper axiomatization of geometry, literally anything can stand in for primitives of theory ("tables, chairs, and beer mugs" for "points, straight lines, and planes")

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Paul Bernays (1967, 497):

> *A main feature of Hilbert's axiomatization of geometry is that the axiomatic method is presented and practised int he spirit of the abstract conception of mathematics that arose at the end of the nineteenth century and which has generally been adopted in modern mathematics. It consists in abstracting from the intuitive meaning of the terms... and in understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation... for which the axioms are satisfied. Thus, an axiom system regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure... [On this conception of axiomatics... logical reasoning on the basis of the axioms is used not merely as a means of asserting intuitions in the study of spatial figures; rather logical dependencies are considered for their own sake, and it is insisted that in reasoning we should rely only on those properties of a figure that either are explicitly assumed or follow logically from the assumptions and axioms.*

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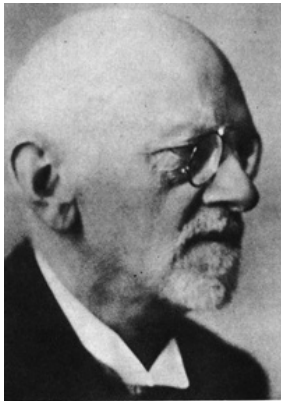The Hilbert programme and creativity

Satisfiability of axioms

- axioms are 'satisfiable' if we can give a model of them ($\Rightarrow$ model theory)

- axioms are independent of we can give models in which one of the axioms is false but all the others hold

- domains of 'points', 'lines', etc in models "are sets of numbers, sets of pairs of numbers, or sets of sets of numbers. Not quite tables, chairs, and beer mugs, but in the same spirit." (153)

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The Frege-Hilbert correspondence

- Frege to Hilbert (27 December 1899): definitions should give meanings of terms (in terms of what is already known), i.e., fix their reference, axioms should express truths

⇒ "simple dilemma: if the terms in the proposed axioms do not have meaning beforehand, then the statements cannot be true (or false), and thus they cannot be axioms. If they do have meaning beforehand, then the axioms cannot be definitions" (155)

- but: Hilbert offered 'implicit' or 'functional' definitions of basic terms; these definitions attempt to capture a structure

- Hilbert to Frege (29 December): purpose of *Grundlagen* is to explore logical relations among principles, defends implicit definitions

- rejected Frege's claim that consistency of axioms is guaranteed by their truth and hence not in need of being established

- for Hilbert, consistency acts as somewhat of a necessary and sufficient condition for something to qualify as mathematics

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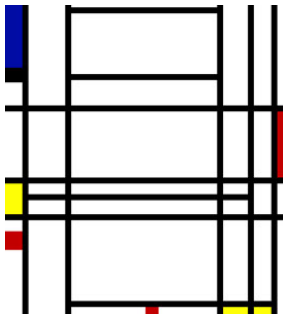## Hilbert to Frege, 7 November 1903

Hilbert

[T]he most important gap in the traditional structure of logic is the assumption... that a concept is already there if one can state of any object whether or not it falls under it... [Instead, what[is decisive is that the axioms that define the concept are free from contradiction... [A] concept can be fixed logically only by its relations to other concepts. These relations, formulated in certain statements I call axioms, thus arriving a view that axioms... are the definitions of the concepts.

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Finitism: the Hilbert programme

- Hilbert: "mathematical analysis is a symphony of the infinite"
- problem: Cantor's set theory—his account of the infinite—was plagued by antinomies
- ⇒ 'Hilbert's programme':

  The goal of my theory is to establish once and for all the certitude of mathematical methods... ("Über das Unendliche" 1925, 184) There is... a completely satisfactory way of avoiding the paradoxes without betraying our science. The desires and attitudes which helps us find this way... are these: (1)... [W]e will carefully investigate fruitful definitions and deductive methods... No one shall drive us out of the paradise which Cantor has created for us. (2) We must establish throughout mathematics the same certitude for our deductions as exists in ordinary elementary number theory, which no one doubts and where contradictions and paradoxes arise only through our own carelessness. (ibid., 191)

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Hilbert's programme

$\Rightarrow$ blend of deductivism, term formalism, and game formalism

- goal: formalize each branch of mathematics (cum logic), proof consistency of each formal system

- vantage point: finitary arithmetic

- not meaningless rules, assertions meaningful, have subject-matter

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Finitary arithmetic

- reference only to specific natural numbers, with effectively decidable properties and relations

- effectively decidable: $\exists$ algorithm which gives answer in finitely many steps

- only bounded—finitary—quantifiers permitted ('there is a prime number between 100 and 100!'), but not unbounded operators, i.e., operators which range over all $\mathbb{N}$ ('there is a prime number greater than 100')

- sentences with with only bounded operators are effectively decidable, but sentence with unbounded properties are not

- sentences such as '$\mathfrak{a} + 1 = 1 + \mathfrak{a}$' and the commutative law '$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$' are finitary because its instances are finitary statements

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- but they are "from our finitary perspective incapable of negation" (ibid. 194)

- reason: negation would assert that there is an instance for which the statement is false, i.e., it contains an unbounded quantifier and hence fails to be finitary

- Hilbert takes Kantian perspective on content of finitary arithmetic:

  Kant taught... that mathematics treats a subject matter which is given independently of logic. Mathematics, therefore, can never be grounded solely on logic. Consequently, Frege's and Dedekind's attempts to do so were doomed to failure. (ibid., 192)

- natural numbers as primitively satisfied preconditions of human thought, are identified with intuitively recognizable 'numerical symbols': |, ||, |||, ||||, ...

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$\Rightarrow$ affinity to term formalism, but careful: symbols denote themselves, i.e., they denote!

- characters more like abstract types, not physical tokens; types are "intuited as directly experienced prior to all experience"

- finitary arithmetic essential to human thought; in fact, nothing is more epistemically secure than finitary arithmetic

- but it needs to be extended, e.g. to cover statements involving unbounded quantifiers, to real and complex analysis, geometry, set theory, etc

$\Rightarrow$ ideal mathematics; but this needs to be treated instrumentally, and formally as in game formalism:

[W]e conceive mathematics to be a stock of two kinds of formulas: first, those to which the meaningful communciations of finitary statements correspond; and, secondly, other formulas which signify nothing and which are the ideal structures of our theory. (ibid., 196)

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An important constraint on ideal mathematics

Definition (Consistency)

"Let us say that the formalized theory T is consistent if it is not possible to derive a contradictory formula, like '$0 = 0$ and $0 \neq 0$', using the axioms and rules of T." (Shapiro, 163)

- constraint on ideal mathematics: formal system must be consistent, and consistent with finitary arithmetic (as in deductivism)
- statement that T is consistent for a formalized axiomatic system T is itself finitary
- $\Rightarrow$ goal of Hilbert programme: provide finitary proofs of consistency of formalized mathematical theories
- $\Rightarrow$ "If T is a formalization of Cantorian set theory, then once we have a finitary consistency proof, we know with maximal certainty that we will not be driven from the paradise." (165)

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Von Neuman's summary of Hilbert's programme

(from Shapiro, 165)

1. To enumerate all the symbols used in mathematics and logic...

2. To characterize unambiguously all the combinations of these symbols which represent statements classified as 'meaningful' in classical mathematics. These combinations are called 'formulas'...

3. To supply a construction procedure which enables us to construct successively all the formulas which correspond to the 'provable' statements of classical mathematics. This procedure, accordingly, is called 'proving.

4. To show (in a finitary... way) that those formulas which correspond to statements of classical mathematics which can be checked by finitary arithmetical methods can be proved... by the process described in (3) if and only if the check of the corresponding statement shows it to be true.

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(from Detlefsen, §5.8)

- Hilbert advocated the Principle of Permanence
- Axiom of Solvability not only constrains Permanence, but also secures adherence to it (we introduce $\sqrt{-1}$ because it helps preserving a basic law—the Fundamental Theorem of Algebra)
- programme contains an essential creative element in its methodology:

Creative Principle

"Having arrived at a certain point in the development of a theory, I may designate (*bezeichnen*) a further proposition as correct (*richtig*) as soon as it is recognized (*erkannt*) that its introduction results in no contradiction with propositions previously admitted as correct... [This is] the creative principle which, in its freest use, justifies us in introducing ever newer concept-formations (*Begriffsbildungen*), the only restriction being that we avoid contradiction." (Hilbert 1905, 135f, as cited by Detlefsen, 290)

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Two basic constraints on programme

Any extension of finitary arithmetic must be

1. consistent, and

2. fruitful.

Furthermore, constructed concepts should be 'constitutively complete' in the sense that the axioms governing them constitute their 'content'.

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Crucial difference between the finitary and the ideal

- Hilbert's programme "allows signs and sign-complexes to play significant roles in mathematical reasoning independently of any interpretation that might be given to them... There is no preaxiomatic grasp or understanding that the axioms are intended to capture, and there is no extra-axiomatic model or structure that might be consulted in a search for new axioms or to correct current axioms." (Detlefsen, 296)

- contrast this with Hilbert's view on finitary arithmetic

⇒ Hilbert's programme is primarily view about mathematical reasoning beyond the primitively given and about the introduction of concepts necessary to succeed to construct consistent and fruitful theories from that base.

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"Wir müssen wissen. Wir werden wissen"
From Wikipedia (http://en.wikipedia.org/wiki/David_Hilbert)

The day before Hilbert pronounced these phrases at the 1930 annual meeting of the Society of German Scientists and Physicians, Kurt Gödel—in a roundtable discussion during the Conference on Epistemology held jointly with the Society meetings—tentatively announced the first expression of his incompleteness theorem.

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Kurt Gödel (1906-1978)

- born in Brno
- PhD Vienna 1930 (proving completeness of first-order predicate calculus)
- 1931 incompleteness
- mid-30s: work on axiom of choice and continuum hypothesis
- 1940 trans-Siberian travel to Princeton
- 1949 Gödel spacetime
- greatest logician since Aristotle

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Gödel's incompleteness theorems

Theorem (Gödel's first incompleteness theorem (informal))

Let T be a formal system. If

- *T contains some arithmetic, and*
- *it can be determined algorithmically whether a given sequence of characters is a well-formed formula and a valid deduction in T,*

then "there is a sentence G in the language of T such that

1. *if T is consistent, then G is not a theorem of T, and*

2. *if T has a property a bit stronger than consistency... then the negation of G is not a theorem of T." (Shapiro, 166)*

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And its consequences

- G is finitary statement meaning, roughly, that G is not provable in T

$\Rightarrow$ If T is consistent, then G is true but not provable.

$\Rightarrow$ There can't be a single formal system for all of ideal mathematics in which all its truths can be derived.

- Arguably, however, Hilbert's programme didn't include a claim to the contrary.

- But there is a second theorem arising when we consider the reasoning behind the first theorem to be reproduced within the system T...

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Gödel's second incompleteness theorem

Theorem (Gödel's second incompleteness theorem (informal))

*"[I]f the formalization of 'provable in T' meets some straightforward
requirements, then we can derive, in T, a sentence that expresses
the following:*

If T is consistent, then G is not derivable in T.

*But, as noted above, 'G is not derivable in T' is equivalent to G. So,
we can derive, in T, a sentence to the effect that*

If T is consistent then G.

*Assume that T is consistent, and that we can derive, in T, the
requisite statement that T is consistent; then it would follow that we
can derive G in T. This contradicts the [first] incompleteness
theorem. So if T is consistent, then one cannot derive in T the
requisite statement that T is consistent." (166f)*

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- Informally, no consistent theory can prove its own consistency (if it contains some arithmetic), and that spells trouble for Hilbert's programme.

- *PA*: formalization of (ideal) arithmetic

- Hilbert requries a finitary proof of *PA*'s consistency

⇒ If *PA* is consistent, then this cannot be proven in *PA*, let alone in its finitary part.

- consistency of system cannot be proved by means of proof weaker than the those of the system itself

⇒ must go beyond framework of finitary mathematics to establish consistency of classical mathematics

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Options for a post-Gödel defence of a Hilbert-style programme

1. challenge formalization of consistency as used in proof of second incompleteness theorem ⇒ what is consistency?

2. finitary arithmetic is inherently informal, i.e., methods of proof of any formalized system cannot include all finitary methods of proof

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In the aftermath of Gödel's results

- Haskell Curry (and Michael Detlefsen): philosophical advocate of formalism; alive in mathematics (e.g. in the group 'Nicolas Bourbaki' in France)

- Curry: move to assertions about formal systems, rather than within them

- mathematics as science of formal systems

- proof of consistency neither necessary nor sufficient for acceptability:

It is obviously not sufficient. As to necessity, so as no inconsistency is known, a consistency proof, although it leads to our knowledge about the system, does not alter its usefulness. Even if an inconsistency is discovered this does not mean complete abandonment of the theory, but its modification and refinement... The peculiar position of Hilbert in regard to consistency is thus no part of the formalist conception of mathematics (Curry 1954; cited after Shapiro, 170)