Intuitionism

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Intuitionism, a first approach

Characterization (Intuitionism)

Intuitionism in or about mathematics is a revisionary form of mathematical constructivism which considers the essence of mathematics to be exhausted by our constructive mental activity, i.e., mental constructions governed by self-evident laws, rather than be constituted by the analytical reasoning involving the formal manipulation of linguistic characters, perhaps revealing the existence and the constitution of independently existing mathematical objects and enabling and governing their application.

The basic revision required

- main revision: reject the law of excluded middle (LEM), i.e., the proposition that it either is or is not the case that Φ, for any proposition Φ (symbolically, $\Phi \vee \neg\Phi$)
- Intuitionists reject LEM because it presupposes, in their view, the independent existence of mathematical objects or the belief that propositions involving them are true or false independently of, and prior to, the mathematician.

$\Rightarrow$ LEM as a consequence of realism (either in ontology or truth value)

- in semantics, we have the related principle of bivalence according to which every proposition is either true or false
- Question: when are LEM and the principle of bivalence equivalent?
- general use of the term 'intuitionism' to cover all philosophies that demur from LEM

$\Rightarrow$ classical logic and classical mathematics vs. the weaker intuitionistic logic and the corresponding intuitionistic mathematics

Classical vs. intuitionistic logic

Consider the following propositions:

(p) 'Not all elements of a set have a certain property P', formally $\neg\forall x Px$.

(q) 'There is an element which lacks property P', formally $\exists x \neg Px$.

Intuitionistic logic (IL):

- content of p is that it is refutable that one can find a mental construction showing that P holds of all elements
- content of q is that one can construct an element of the set and show that it does not exemplify P
- Clearly, $q \rightarrow p$, but not the converse because it's "possible to show that a property cannot hold universally without constructing a number for which it fails." (9)

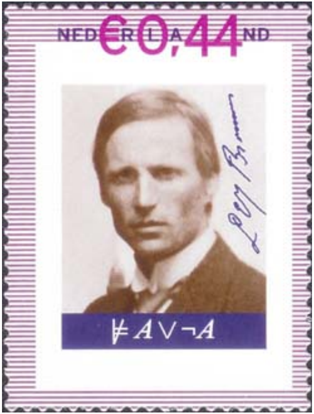
Classical logic (CL):

- content of p and q realistically given

$\Rightarrow$ $p \leftrightarrow q$

- Similarly, all laws of classical logic which rely on LEM are rejected in IL.
- Example: law of double negation elimination in CL, which states that for any proposition Φ, $\neg\neg\Phi \rightarrow \Phi$
- IL: $\Phi \rightarrow \neg\neg\Phi$, but $\neg\neg\Phi \not\rightarrow \Phi$
- To see this, suppose that one can construct a contradiction from $\neg\Phi$, in which case a reductio inference (admissible both in CL and IL) yields $\neg\neg\Phi$.
- CL: by double negation elimination, one can also infer Φ
- IL: this is unwarranted, unless you have independent reason for, i.e., a mental construction of, Φ (e.g., to derive a contradiction from $\neg\exists n Pn$ does not amount to a construction of a number n such that Pn)

Luitzen Egbertus Jan Brouwer (1881-1966)



- Dutch mathematician and philosopher
- PhD 1907 U Amsterdam
- best known for his topological fixed-point theorem
- 1912 Extraordinarius at U Amsterdam
- worked in topology, set theory, measure theory, complex analysis, and of course foundations
- never lectured on topology—only on foundations of intuitionism
- temporarily kept his iconoclast attitude under wraps, until 1912 (coincidence?): inaugural lecture, *Intuitionisme en formalisme*

Brouwer's intuitionism

Carl Posy, "Intuitionism and philosophy", in Shapiro (ed.), *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford University Press (2005), 318-355.

- of interest to philosophers "because his mathematics rests upon a unique epistemology, a special ontology, and an underlying picture of intuitive mathematical consciousness." (Posy, 319)
- Shapiro: strong Kantian background in Brouwer, recognizes Kant as precursor
- Brouwer developed intuitionism in response to Hilbert's dominating formalism
- (practice of) finitary arithmetic: not much difference between intuitionism and formalism
- but they differ regarding the source of the exact validity of maths; for the intuitionist, it's the human intellect, while for formalism, only on paper

- mathematical truths not analytic (i.e., true by virtue of meaning), but synthetic (like Kant) and a priori (against Mill)
- antirealism in ontology and truth value (he considered realism incorrect and outdated)
- maths is mind-dependent, its essence is idealized mental construction
- maths concerns active role of mind in structuring human experience
- but Brouwer rejected Kantian view of geometry, particularly his view of space
- instead: "proposal to found all of mathematics on a Kantian view of time" (Shapiro, 176) (instead of geometry on the forms of our spatial perception)

- i.e., base natural numbers on 'forms' of temporal perception when we apprehend world as linearly ordered sequence of distinct moments
- discrete moments give rise to natural numbers; but since our intuition "also unites the 'continuous and discrete' and 'gives rise immediately to the intuition of the linear continuum' " (177), we also obtain the rational and the real numbers
- then do analytic geometry based on $\mathbb{R}$

$\Rightarrow$ even geometry is ultimately grounded in our temporal perceptions "practice of mathematics flows from introspection of one's mind" (180):

> *The... point of view that there are no non-experienced truths... has found acceptance with regard to mathematics much later than with regard to practical life and to science. Mathematics rigorously treated from this point of view, including deducing theorems exclusively by means of introspective construction, is called intuitionistic mathematics. (Brouwer 1948, 90; cited after Shapiro, 180)*

$\Rightarrow$ in maths, to be is to be constructed

- incorrect to believe in existence of unknowable truths

The problem of the continuum

- right before Brouwer enters the stage, infinity came of age in mathematics (Weierstrass, Dedekind, Cantor)
- Brouwer: there is a problem at the heart of these accounts of infinity...

Problem of the continuum

"[T]he new set-theoretic penchant toward arbitrary sets and sequences; sets that cannot be described and sequences whose elements cannot be calculated." (Posy 322) So what was required was a novel account of the mathematical continuum on an epistemically sound basis.

On the mathematical side [Brouwer] provided a constructive theory of finite mathematics together with a new set theory encompassing both discrete and continuous infinities. And philosophically he anchored this with special epistemological and ontological doctrines, doctrines that themselves derived from an overall phenomenological outlook. (Posy, 322)

The constructive conception of mathematics and LEM

- Brouwer constructive conception of mathematics $\Rightarrow$ rejection of LEM
- (cf. also my example in the parenthesis at the end of p. 5)
- Let P be a property of natural numbers and Φ be a proposition asserting the existence of a natural number n which exemplifies P: $\exists n \in \mathbb{N}(Pn)$
- So LEM would entail that either there is or is not a number $n \in \mathbb{N}$ such that Pn.

$\Rightarrow$ On Brouwer's conception this would be established if we either construct a number $n \in \mathbb{N}$ which is P, or derive a contradiction from the assumption that there is such a number.

- However, before we haven't done either, we couldn't claim this instance of LEM, i.e., we couldn't assert $\exists nPn \vee \neg\exists nPn$.

$\Rightarrow$ Generally, we cannot presuppose LEM.

- LEM amounts to an inacceptable sort of omniscience (every proposition can be proved or be reduced to absurdity)
- Brouwer also rejected the use of impredicative definitions as we cannot construct an object from a pre-existing collection containing that entity to be constructed
- Shapiro: "Brouwer's conception of the nature of mathematics and its objects leads to theorems that are (demonstrably) false in classical mathematics." (181), and vice versa
- Example: it implied that functions from $\mathbb{R}$ to $\mathbb{R}$ are, by necessity, uniformly continuous (i.e., there are no discontinuous functions on $\mathbb{R}$ as they would entail instances of LEM!)
- The resulting mathematics is quite radically different from its classical counterpart (which have some limited use in the sciences).

$\Rightarrow$ divorce between maths and empirical science

Separable mathematics

- separable = discrete maths
- produced by predicative sequential process, parallel to Cantor's theory up to $\aleph_0$
- conflict with Cantor's theory of higher cardinals
- (because he accepts reductio proofs) Brouwer has no bones with Cantor's proof that the "species" (roughly, set) of all (constructible) infinite sets of naturals is not countable
- but we haven't thereby constructed a cardinal greater than $\aleph_0$, so this species "denumerably unfinished"

The continuum, set theory, and full mathematics

- while at the countable level, Brouwer stays very close to Cantor, he departs from classical mathematics, however, in his theory of the continuum
- continuum built up from "choice sequences" which may or may not be fully determinate
- set theory: species (roughly, sets), spreads (constructive sets of sequences), fans (= "a spread with only finitely many successors to each admissible finite sequence", 325)
- emerging intuitionistic mathematics (a) sometimes weaker than, (b) sometimes refinement of, (c) sometimes inconsistent with, classical mathematics

Intuitionistic vs. classical mathematics

(a) weaker: trichotomy that any real number is either less than, equal to, or greater than any other given real number is not true intuitionistically

(b) refinement: e.g., there are different ways in which a real number can fail to be rational, either it is not rational (as far as we know), or it cannot be rational (i.e., we have a constructive proof to this effect)

(c) deviation: Brouwer's notorious Continuity Theorem in intuitionistic maths, which states that "[f]ully defined functions of real numbers are always continuous" (326)

Actually this theorems comes in two forms ((2) $\rightarrow$ (1), but (1) $\not\rightarrow$ (2)):

1. weak form: "a function cannot be both total (i.e., have a value at every point in its domain) and discontinuous at some point in its domain" (ibid.)
2. strong form: "every function that is total on $\mathbb{R}_{[0,1]}$ is continuous." (ibid.)

Brouwer's philosophy:
The phenomenology of intuition and constructivity

General phenomenology:

- individual elements of "primordial consciousness"
- we discern an order among them (temporal)

$\Rightarrow$ "attenuated mental sequences", which form "building blocks of the subject's awareness of ordinary empirical objects

- so mind produces sequences and correlates them, willfully imposing structure on "roil of consciousness"

Phenomenology of mathematics:

- "first act of intuitionism": breaking up of one primordial element into two and abstracting away from all particular two-ities to get "empty form of common substratum of two-ities"; forms phenomenological basis of arithmetical identities in particular, gives discrete infinity and the continuum
- "second act of intuitionism": generates infinite entities, either through choice sequences, or more abstractly by means of equivalence relations, enables more abstract constructions

Epistemology, intuition, and ontology

- Brouwer (like Kant): mathematical knowledge is synthetic a priori
- a priori both in that it is non-empirical (because it arises differently from sensory knowledge) and necessary
- mathematical intuition is temporal and abstract, i.e. it "far outstrips any sort of sensory grasp" (Posy, 332)
- every mathematical object there is must have been constructed, i.e. existence is tied to constructibility

$\Rightarrow$ against indirect existence proofs which rely on LEM

- (Hence, he accepts Cantor's proof that $\mathbb{R}$ is uncountable without accepting the existence of an unconstructed uncountable cardinal number)

Negative doctrines

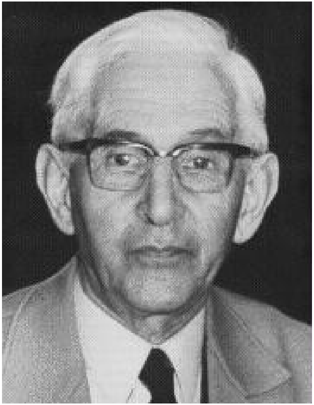
- Intolerance of classical mathematics: mathematical truths are necessary, so since classical mathematics contradicts some of these, it can't be accepted on pain of inconsistency
- So classical maths in fully meaningful, but necessarily false.
- Logic: LEM only valid at finitary stage; given the incompleteness of infinite mathematics, however, LEM cannot be valid there
- Importantly: logic follows ontology, but cannot lead!
- Language: see next slide

The linguistic turn in foundations

- logicism and formalism exhibit a focus on logic and language of maths
- Alberto Coffa (1991): "semantic tradition"; Michael Dummett (much earlier): "linguistic turn"
- Brouwer resisted that tendency, as he considered language as only the imperfect medium for communicating our mental constructions and logic as being subsidiary to mathematics, encoding the basic self-evident rules of our mental constructions

in... construction... neither the ordinary language nor any symbolic language can have any other rôle that that of serving as a non-mathematical auxiliary, to assist the mathematical memory or to enable different individuals to build up the same [construction]. For this reason the intuitionist can never feel assured of the exactness of a mathematical theory by such guarantees as the proof of its being non-contradictory, the possibility of defining its concepts by a finite number of words... or the practical certainty that it will never lead to a misunderstanding in human relations. (1912, 81; cited after Shapiro, 185)

Arend Heyting (1898-1980)



- Dutch mathematician and logician
- PhD 1925 Amsterdam (under Brouwer)
- secondary teacher at Enschede 1925-1936
- 1930 Erkenntnis Symposium at Königsberg: three-way duel between Heyting (intuitionism), Carnap (logicism), von Neumann (formalism)
- appointment 1936 to U Amsterdam
- axiomatizations of constructive theories (geometry)
- formalisation of intuitionistic logic, algebra, Hilbert spaces, proof theory etc
- Heyting algebra, Heyting arithmetic

Heyting's anti-metaphysical stance

According to Heyting, using LEM amounts to invoke illicit metaphysical arguments:

> *If 'to exist' does not mean 'to be constructed', it must have some metaphysical meaning... We have no objection against a mathematician privately admitting any metaphysical meaning he likes, but Brouwer's programme entails that we study mathematics as something simpler, more immediate than metaphysics. In the study of mental mathematical constructions 'to exist' must be synonymous with 'to be constructed'. (Heyting 1956, 2; cited after Shapiro, 188)*

Heyting sometimes went as far as claiming that mathematics is empirical (although not along Millean lines):

> *The affirmation of a proposition is not itself a proposition; it is the determination of an empirical fact, viz., the fulfillment of the intention expressed by the proposition. (1931, 59; Shapiro, ibid.)*
>
> *Intuitionistic mathematics consists... in mental constructions; a mathematical theorem expresses a purely empirical fact, namely the success of a certain construction. (1956, 8; ibid.)*

- But: Brouwer's own Kantianism is not metaphysically neutral!

Heytingean intuitionism

- developed a rigorous axiomatization of intuitionistic logic, Heyting predicate calculus
- Heyting: language of CL (and maths based on it) best understood in terms of objective truth conditions, language of IL (and maths based on it) in terms of proof conditions
- knowing a proposition Φ means to produce a constructive proof of it
- Heyting (1930) gives a semantics from which it is clear that many instances of LEM are inadmissible
- 'Heyting semantics' (Shapiro), or often 'Brouwer-Heyting-Kolmogorov interpretation', or 'BHK interpretation', which is supposed to capture the meaning of the logical symbols in the language in which IL is cast

Intuitionistic logic

for details, cf. Posy 336f

- IL: full language of first-order logic, includes ten axiom schemata
- IL contains three inference rules: modus ponens, $\forall$-introduction, $\exists$-elimination
- larger number necessary because logical particles cannot be interdefined through the DeMorgan equivalences, so each symbol comes separately with an axiom
- but first nine axiom schemata are equivalent to CL-minus-LEM, the axiom schemata

$$\neg\neg\Phi \rightarrow \Phi$$

  in CL is replaced in IL by

$$\Phi \rightarrow (\neg\Phi \rightarrow \Psi).$$

$\Rightarrow$ syntactically intuitionistic logic is proper subsystem of CL

- add (standard) axioms for identity and arithmetic $\Rightarrow$ formal system HA for intuitionistic arithmetic ('Heyting arithmetic'), which is a proper subsystem of the standard PA
- among semantic properties of IL, philosophically most intriguing is model-theoretic approach to semantics

$\Rightarrow$ "precise semantic notion of truth and logical validity" (Posy, 338)

- Evert Beth (1947), Saul Kripke (1965): not single model, but collection of 'nodes', partially ordered by 'accessibility' relation (demonstrably faithful to IL)

A model-theoretic counterexample to LEM

- model structure contains three nodes, w_1, w_2, w_3 such that w_2 and w_3 are accessible to w_1 though not to one another
- In Kripke models, e.g.:
  1. Φ is true at node w only if Φ is true at any node w' accessible to w
  2. $\neg\Phi$ is true at node w only if Φ is not true at any node w' accessible to w
  3. $(\Phi \vee \Psi)$ is true at w only if either Φ is true at w or Ψ is
- Suppose Φ is true at w_2 but not at w_3.

$\Rightarrow$ Φ is not true at w_1 (from (1) and Aw_3w_1)

$\Rightarrow$ $\neg\Phi$ is not true at w_1 (from (2) and Aw_2w_1)

$\Rightarrow$ $\Phi \vee \neg\Phi$ is not true at w_1 (from (3) and previous lines)

$\Rightarrow$ counterexample to LEM

Heyting's interpretation

- Heyting: not that IL is formal statement of intuitionistic ontology
- instead: special intuitionistic meaning of logical particles
- intuitionism: maths has no unknowable truths

$\Rightarrow$ in maths: to be true is to be provable; IL results from applying this idea to semantics of connectives and quantifiers

- 'truth' is replaced by 'assertability' (or 'provability')
- set out conditions of provability of propositions, as follows...

BHK interpretation

Brouwer-Heyting-Kolmogorov interpretation

(Cited from Shapiro, 186)

- [$\perp$ is not provable.]
- A proof of a sentence of the form 'Φ and Ψ' consists of a proof of Φ and a proof of Ψ.
- A proof of a sentence of the form 'either Φ or Ψ' consists of either a proof of Φ or a proof of Ψ.
- A proof of a sentence of the form 'if Φ then Ψ' consists of a method for transforming a proof of Φ into a proof of Ψ.
- A proof of a sentence of the form 'not-Φ' consists of a procedure for transforming any proof of Φ into a proof of absurdity. In other words, a proof of not-Φ is a proof [that] there can be no proof of Φ.
- A proof of a sentence of the form 'for all $x, \Phi(x)$' consists of a procedure that, given any n, produces a proof of the corresponding sentence $\Phi(n)$.
- A proof of the sentence of the form 'there is an x such that $\Phi(x)$' consists of the construction of an item n and a proof of the corresponding $\Phi(n)$.

Some remarks

- 'not-Φ' is equivalent to 'if Φ then $\perp$'
- one can't prove an existentially quantified sentence without showing how to construct such an x
- BHK interpretation is informal since 'construction' is not defined and thus open to different interpretations
- It follows immediately (already at informal level) that LEM is not generally true: to prove 'Φ or not-Φ' one must either deliver a proof of Φ or give a procedure for transforming any proof of Φ into a proof of absurdity.
- In case we haven't done either, the statement '$\Phi \vee \neg\Phi$' does not hold.
- Once we have done either, the particular instance of 'tertium non datur' is established.

Heyting, formalism, and language

- ironic that Heyting took such a formal approach, which seems anathema to Brouwer's attitude towards language and formalism (according to `http://en.wikipedia.org/wiki/Arend_Heyting`, Brouwer called Heyting's work a "sterile exercise")
- but he generally shared Brouwer's take on language and logic; in fact, Heyting considered mathematics to be prior to logic, the task of which it is to try to formally capture legitimate mathematical constructions, not to codify it

⇒ Heyting's axiomatic work brought intuitionistic mathematics under purview of proof theory

- typically, classical logic is used to study formal systems which employ intuitionistic logic

⇒ detailed study of role of LEM in mathematics, opened space for alternative logics

Sir Michael Anthony Eardley Dummett (1925-2011)

- British analytic philosopher
- BA in PPE, Christ Church College, Oxford 1950
- Fellow at All Souls College, Oxford 1950-1979, Wykeham Professor of Logic at Oxford 1979-1992
- philosophy of language, mathematics, logic, Wittgenstein, Frege
- anti-racism activist
- *Frege: Philosophy of Language* (1973), *Elements of Intuitionism* (1977), *Truth and Other Enigmas* (1978), *Frege: Philosophy of Mathematics* (1991)

Dummettian intuitionism: meaning

Dummett's vantage point is language, since considerations regarding which logic is correct turn on meaning, which is determined by use, by its role in logical inference:

> *The meaning of a mathematical statement determines and is exhaustively determined by its use... if two individuals agree completely about the use to be made of [a] statement, then they agree agree about its meaning... An individual cannot communicate what he cannot be observed to communicate." (Dummett 1973, 98f; cited after Shapiro, 190)*

The manifestation requirement

Thesis (Manifestation requirement)

"[A]nyone who understands the meaning of an expression must be able to demonstrate that understanding through her behaviour—through her use of the expression." (191)

$\Rightarrow$ understanding not ineffable: one understands expressions iff one knows how to use them correctly

- In contrast, for Frege one understands a sentence if one 'grasps' its 'sense'.
- Dummett: against Frege's claims that these 'senses' are objective (= mind-independent) entities—they're way too private to play any role in mathematical practice

$\Rightarrow$ stark contrast to Brouwer and Heyting

From manifestation to intuitionism

- Question: shouldn't mathematicians' use of LEM license it?

$\Rightarrow$ needs semantics to fix 'legitimate' or 'correct' use

- For Dummett, in classical mathematics,

the central notion is that of truth: a grasp of the meaning of a sentence... consists in a knowledge of what it is for that sentence to be true. Since, in general, the sentences of the language will not be the ones whose truth-value we are capable of effectively deciding, the condition for the truth of such a sentence will be one which we are not, in general, capable of recognising as obtaining whenever it obtains, or of getting ourselves into a position in which we can so recognise it. (1973, 105; cited after Shapiro, 193)

- $\Rightarrow$ replace truth with verifiability or assertability as main constituent of semantics
- $\Rightarrow$ instead of truth conditions for formulae, we seek proof conditions for them
- against meaning holism: at least some parts of language can be understood independently of others
- particularly applies to logical terminology including connectives and quantifiers
- $\Rightarrow$ offers basis on which practice can be criticized: ways logical operators are usually introduced into proofs inconsistent with classical logic
- In particular: rules for introducing negation and disjunction separately do not warrant LEM when combined

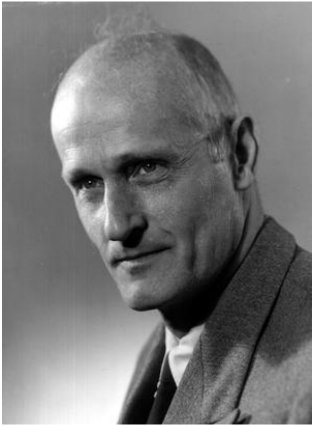
Dummett's global semantic anti-realism

- classical bivalent semantics: truth and knowability separate
- Dummett's global semantic anti-realism: all truths are knowable in principle, possibility of unknowable truth ruled out on a priori grounds
- This also means that if humans were capable of deciding truth values of every formula, then all instances of LEM would be sanctioned by this capability and intuitionistic logic would reduce to classical logic (even under BHK semantics).
- Dummett's argument could be resisted on two grounds:
  1. show that there is a semantics which meets the manifestation criterion (and others) and sanctions classical logic
  2. reject Dummett's framework since classical mathematics does not stand in need of the justification Dummett demands in these criteria

Dummett's radical generalization

- ⇒ radically generalizes assertabilist semantics to apply to all human language, not just mathematics
- ⇒ IL, and not CL, will be appropriate logic not just for mathematics, but for any discourse in which there are undecidable propositions

Stephen Cole Kleene (1909-1994)



- American mathematician
- PhD 1934 Princeton (under Alonso Church)
- 1935-41 U Wisconsin Madison, 1941-42 Amherst College, 1946-79 U Wisconsin Madison
- developed recursion theory (with Church, Gödel, Turing, Post), mathematical intuitionism
- tried to connect computability of recursive functions with constructive proofs required by intuitionism (constructive proofs are those that can be 'realized' by computation of numbers)

Taking stock

Credits:

- rich technical studies, mature constructivism reflecting human aspect of mathematics

Debits:

1. intuitionism is a deviation against modern norm, regarded either as "messy distraction" or technically "quixotic curiosity" (Posy, 344)
2. internal dissonance: Brouwer's (or any) metaphysical grounding of logic is poor bedfellow of Heyting-Dummett assertabilism
3. intuitionism depends on existence of undecidable propositions that it "cannot in fact construct, and whose possible existence [it...] may not assert" (345) because if it specified an undecidable proposition, the assertability clause for negation would entail that the proposition is false, thus deciding it!

Carl Posy

So that is where we are in intuitionism: a house internally divided, bent upon an eccentric technical mission, and based on a fundamental assumption which goes against its own internal standards. [...I believe that a] Kantian perspective can serve to reconcile the opposing intuitionistic streams. (345)