Numbers exist—or they don't

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Introduction to Philosophy of Mathematics

Two camps

According to Shapiro, there are two main camp in the "contemporary scene" in philosophy of mathematics:

1. Numbers exist: the axiom of arithmetic that 0 exists and the theorem that $\forall n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $m > n$ and m is prime jointly imply that there are infinitely many numbers, and so numbers exist
    - Often, this existence is taken to be of the same kind as that of physical bodies.
    - Representatives: Plato, Frege, Crispin Wright, Bob Hale
2. Numbers don't exist: accept the importance of mathematics, but insist that it ought not to be read literally
    - This is often taken to accept that mathematics has to be reformulated in a way such as to free it from an ontological commitment mathematical objects.
    - Representatives: intuitionists, Hartry Field, Charles Chihara

Paul Benacerraf, 'Mathematical truth' (1973)

Paul Benacerraf. Mathematical truth. *Journal of Philosophy* 70 (1973): 661-679. Reprinted in Benacerraf and Hilary Putnam (eds.), *Philosophy of Mathematics*, Cambridge University Press ([2]1983).

Dilemma (Benacerraf 1973)

EITHER it is very hard to understand how we can come to know any mathematical truths OR we must give up a semantic continuity of mathematical with everyday and scientific discourse (OR we must give up our usual semantics of ordinary and scientific languages)

Semantic desideratum

"[M]athematical statements should be understood in the same way as ordinary statements, or at least respectable scientific statements. That is, we should try for a uniform semantics that covers ordinary/scientific language as well as mathematical language." (Shapiro, 31)

The first horn: double realism

- appreciate the natural alliance of realism in ontology with realism in truth-value: mathematical statements deal with objective features of the world, and the terms of its language denote (e.g. numerals denote numbers); while logically independent, that mathematical objects really exist in their own right is suggested by the objective truth of mathematical assertions
- realism in truth-value leads to a straightforward satisfaction of the semantic desideratum: if a realism in truth-value holds for scientific (or ordinary) discourse, and our realism guarantees that this is also the case for mathematics, then the desideratum is satisfied
- But if mathematical objects objectively exist, and they are abstract entities without causal nexus to our material world, how can we come to know anything about them?

$\Rightarrow$ 'double realism' nicely satisfies the semantic continuity criterion, but at the expense of a deep epistemological puzzle

The second horn: double antirealism

- idealism in ontology makes it natural to accept an idealism in truth-value: mathematical truth about mental mathematical objects depends on the mind
- similarly for other forms of antirealism: one's ontological position is naturally associated with one's analysis of mathematical truth (e.g. Hartry Fields 1980: nominalism about mathematical objects, combined with an assertion that mathematical statements have vacuous truth-values)

⇒ no (or little) epistemological puzzle: either there is a sense in which there is nothing to 'know', or it's all mental/ideal, in which case there's a straightforward connection in the mind

- However, if one wants to maintain the semantic continuity desideratum, one is committed to denying realism about both ordinary and scientific discourse—an option many found unattractive.

⇒ One gets a straightforward epistemic account, but at the expense of either rejecting the semantic desideratum or accepting antirealism about ordinary and scientific discourse

(1) Gödel

Central essays, both in his *Collected Works*, volume II:

- 'Russell's mathematical logic' (1944)
- 'What is Cantor's continuum problem' (1964)

Gödel on impredicative definitions

Kurt Gödel. Russell's mathematical logic (1944). In his *Collected Works, Volume II*, edited by Solomon Feferman et al., Oxford University Press (1990), 119-141.

- impredicative definitions are permissible, indeed necessary for mathematics
- Russell's 'vicious circle principle' conflicts with classical mathematics:

Gödel 1944:

> *[Russell's vicious circle principle] says that "no totality can contain members definable only in terms of [the whole] totality, or members involving or presupposing this totality"... (125) it is first to be remarked that, corresponding to the phrases "definable only in terms of", 'ìnvolving", and "presupposing", we have have really three different principles, the second and third being much more plausible than the first. It is the first which is of particular interest, because only this one makes impredicative definitions[(footnote suppressed)] impossible and thereby destroys the derivation of mathematics from logic [...] and a good deal of modern mathematics itself. (127)*

Gödel on impredicative definitions

Gödel 1944:

> *It is demonstrable that the formalism of classical mathematics does not satisfy the vicious circle principle in its first form, since the axioms imply the existence of real numbers definable in this formalism only be reference to all real numbers [...]*
>
> *I would consider this rather as a proof that the vicious circle principle is false than that classical mathematics is false [...]*
>
> *[...] it seems that the vicious circle principle in its first form applies only if the entities involved are constructed by ourselves. In this case there must clearly exist a definition (namely the description of the construction) which does not refer to a totality to which the object defined belongs, because the construction of a thing can certainly not be based on a totality of things to which the thing to be constructed belongs. If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members which can be described (i.e., uniquely characterized)[(footnote suppressed)] only be reference to this totality [...] (127f)*

Realism and impredicative definitions

- For a realist in ontology like Gödel, "a definition is not a recipe for constructing an object, but only a method for describing or pointing to an already existing entity." (Shapiro 204)
- "From this perspective, impredicative definitions are innocuous." (ibid.)

Physical and mathematical objects

- Gödel cites approvingly Russell who he takes to claim an analogy between mathematical objects and ordinary physical objects:

Gödel 1944

> *[... Russell] compares the axioms of logic and mathematics with the laws of nature and logical evidence with sense perception, so that the axioms need not necessarily be evident in themselves, but rather their justification lies (exactly as in physics) in the fact that they make it possible for these "sense perceptions" to be deduced; which of course would not exclude that they also have a kind of intrinsic plausibility similar to that in physics. I think that (provided "evidence" is understood in a sufficiently strict sense) this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future. (121)*

$\Rightarrow$ Gödel's controversial suggestion: "just as we build up sophisticated physical theories in order to account for (and predict) sensory observations, in mathematics we build up sophisticated theories to account for 'intuitions', or entrenched beliefs about mathematical objects." (Shapiro 205)

Gödel on mathematical intuition

Kurt Gödel. What is Cantor's continuum problem? (1964). In his *Collected Works, Volume II*, edited by Solomon Feferman et al., Oxford University Press (1990), 254-270.

Gödel 1964:

[...] the objects of transfinite mathematics [...] clearly do not belong to the physical world, and even their indirect connection with physical experience is very loose [...] (267)

But, despite their remoteness from sense experience, we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induced us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics [...]

The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis. (268)

Gödel on mathematical intuition

- Gödel accepts that we may not always have immediate knowledge of mathematical objects themselves—but this is hardly different from contemporary physics.
- Gödel's concept of 'intuition' is Kantian in origin, but unlike Kant's (which makes mathematics mind-dependent), his concept of our mathematical intuitions makes them "glimpses (of sorts) into an objective mathematical realm" (Shapiro 207)

Gödel on Cantor's continuum hypothesis

- As discussed earlier, Gödel's realism suggests that there is an objective fact of the matter as to whether the continuum hypothesis is true or false

⇒ independence from ZFC shows that we need a more powerful a set theory

Gödel 1964:

> *[...] the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality. (260)*

(2) Quine and the web of belief

- Quine offers an empiricist epistemology which is superior to that of Mill
- naturalism as "abandonment of first philosophy"
- Quine: virtually no a priori knowledge
- In many ways, Quine is a reaction to logical empiricism, and particularly to Carnap.

Carnap vs. Quine

- Carnap: mathematics concerns 'framework principles', and is analytic a priori
- Quine: reaction to Carnap in 'Two dogmas of empiricism' (1951)
- First 'dogma': analytic-synthetic distinction
  - ⇒ must be given up in favour of (semantic) holism
- Second 'dogma': 'reductionism', i.e., the idea that each meaningful statement is equivalent to some logical construct upon terms which refer to immediate experience
  - Instead, our total system of beliefs is a 'seamless web' which faces the 'tribunal of experience' together
  - ⇒ confirmation holism

Shapiro reads Quine in a way compatible with holding that some statements are analytic... but this goes against the usual interpretation of Quine (and against what Quine says!).

Empiricism, holism, realism

Hilary Putnam (1971). *Philosophy of Logic*. New York: Harper Torchbooks.

- Quine joins Mill in arguing that mathematics is based on observation, similar in status to the theoretical parts of science.
- His holism (together with his scientific realism) implies realism in truth value—the entire web is approximately true.
- As for realism in ontology, let's consider the indispensability argument as articulated by Quine and Putnam (1971, ch. 5)
- target: nominalism

Position (Nominalism)

There are no abstract objects.

Question:

Can a nominalistic language serve the needs of science?

Answer according to Putnam and Quine:

No.

Physics needs mathematical objects

Putnam (1971, 37)

> *[Newton's law of gravitation] asserts that there is a force f_{ab} exerted by any body a on any other body b. The [...] magnitude F [of the force] is given by:*
>
> $$F = gM_aM_b/d^2$$
>
> *where g is a universal constant, M_a the mass of a, M_b the mass of b. and d the distance which separates a and b.*
>
> *The point of the example is that Newton's law has a content which, although in one sense is perfectly clear (it says the gravitational 'pull' is directly proportional to the masses and obeys an inverse-square law), quite transcends what can be expressed in nominalistic language. Even if the world were simpler than it is, so that gravitation were the only force, and Newton's law held exactly, still it would be impossible to 'do' physics in nominalistic language.*

- Physics (but also other sciences) is full of magnitudes which are quantified by real numbers: volume, velocity, force, temperature, pressure, etc.

The indispensability argument
Colyvan (2012, 43)

Mark Colyvan (2012). *An Introduction to the Philosophy of Mathematics*. Cambridge: Cambridge University Press.

(P1) We ought to have ontological commitment to all and only the entities that are indispensable to our current best scientific theories.

(P2) Mathematical entities are indispensable to our best scientific theories.

(C) We ought to have ontological commitment to mathematical entities.

Remarks (cf. Colyvan 2012, §3.2):

- How should 'indispensability' be understood? Something along the following lines: an entity is dispensable if it is eliminable and the resulting theory is still at least as attractive as the original one.
- (P1) is supported by Quine's naturalism and holism.
- Thus, the argument presupposes that there is only one kind of existence, and that anything over which is quantified in a first-order theory must be ontologically committed to.

Some problems for Quine

1. Quine's account doesn't explain either the necessity of mathematical truth or its being a priori
    - Quine rejects a priori knowledge and is suspicious of modality; still, he would at least have to account for the apparent necessity and a priority of maths.
    - For Quine, mathematics and logic are of course revisable.
2. What is the status of those parts of mathematics which is not connected at all to empirical science, such as the higher reaches of set theory?
    - Quine: these theories are hypothetical; gives them a deductivist treatment.
3. Quine makes the application of mathematics to science a matter of truth, which departs from the usual approach (though that is not a priori a problem).

(3) Maddy's set-theoretic realism

Penelope Maddy (1990). *Realism in Mathematics*. Oxford: Oxford University Press.

- double realism, synthesis between Gödel's realism and Quine's empiricism
- naturalism
- According to Maddy, realism regarding a type of entity is justified if the objective existence of these entities is part of our best explanation of the world.

'Compromise platonism'

- Maddy proposes a 'compromise realism' based on two main ingredients
  1. Putnam-Quine indispensability argument
  2. Gödel's acceptance of purely mathematical forms of evidence and justification

⇒ two-tiered epistemology:
  1. lower level: 'intuition' supporting basic mathematical theories
  2. upper level: 'extrinsic' justification, through application to natural science

⇒ Maddy must come up with a naturalistic account of the 'Gödelian' lower epistemological level

⇒ She must thus accept a responsibility for explaining mathematical intuition—it must be respectable on scientific grounds.

Maddy on the perception of mathematical entities

- objects to be justified: sets (not Sets, i.e., pure sets)
- Maddy: we actually perceive sets directly
- Based on work by the neuropsychologist Donald O Hebb, Maddy claims that our brains have 'object-detectors' and 'set-detectors'.
- Example: perception of 8 shoes vs perception of 4 pairs of shoes
- Based on perceptions like this, we infer Set Theory (with an axiom of infinity), which then provide a uniform foundation for mathematics (which is part of our web of belief—she is partly Quinean).
- concept of 'impure a priori': we need experience to (initially) form concepts, but not afterwards
- Maddy is clearly closer to empiricism than Gödel, but agrees with him that every well-formed sentence of set theory (such as the continuum hypothesis) has a truth value.

Nominalism, antirealism

Position (Nominalism)

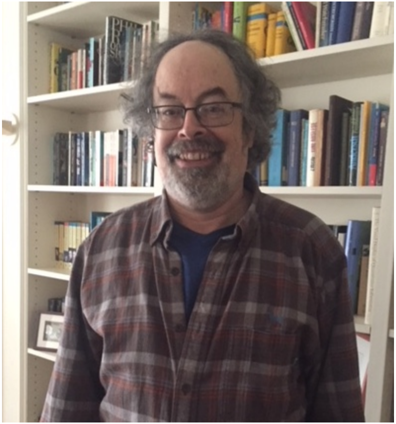
There are no abstract objects such as mathematical objects.

Prominent antirealists include (apart from the intuitionists):

1. Hartry Field: accept mathematical language at face value, but given his antirealism-in-ontology, this means that although propositions involving mathematical terms have objective truth values, these are vacuous (universally quantified statements such as 'All real numbers are prime' are vacuously true, and existentially quantified statements such as 'There exists a prime number larger than 100' are false); but this is not the point of mathematics.
2. Charles Chihara: attempts to give systematic way to interpret the language of mathematics such that it is free of references to mathematical objects (sentences keep their standard truth values, so he is a realist in truth value also).

(1) Hartry Field's fictionalism

Hartry Field (1980). *Science Without Numbers*. Princeton: Princeton University Press.



- Field: the only serious argument for realism: Putnam-Quine indispensability argument

$\Rightarrow$ antirealism needs to undermine this argument

Field's focus: real analysis (RA) and physics

In this example, the indispensability argument takes the following form (Shapiro 2000, 228):

1. RA refers to, and has variables that range over, abstract objects called 'real numbers'. Moreover, one who accepts the truth of the axioms of RA is committed to the existence of these abstract entities.
2. RA is indispensable for physics. That is, modern physics can be neither formulated nor practised without statements of RA.
3. If RA is indispensable for physics, then one who accepts physics as true of the material reality is thereby committed to the truth of RA.
4. Physics is true, or nearly true.
5. Therefore, real numbers exist.

Thus, we get to realism in ontology.

- Field accepts (1), (2), and (4), but denies (2).
- RA may be a practical necessity, but that doesn't make it essential to the scientific practice in an ontological sense.
- Field: science can be done without numbers (hence the title of his book)

⇒ find a nominalistic language for science (or at least for physics)

⇒ first order of business: rebut Putnam's claim that this is hopeless

Field's nominalistic language

- Field develops a nominalistic version of Newtonian gravitation theory, as an example.
- His formulation ranges over spacetime points and regions—these exist concretely, they are not mathematical objects:
  1. The cardinality and geometry (structure) of collections of spacetime points depend on a physical theory, not on mathematics.
  2. Contingent properties of spacetime points (such as the strength of gravitational field or the temperature at points) are essential parts of causal explanations of observable phenomena.
- This leads us to the debate between substantivalism and relationalism. Field assumes substantivalism.
- But, Shapiro says, spacetime points are very different from typical physical objects: they don't endure through time, they can't be moved in space, they cannot be decomposed or destroyed, they have neither mass nor extension, and they exist necessarily (though this last point is questionable).

Field's nominalistic reconstruction

- Field includes primitive physical relations among spacetime points, such as Betweenness, etc.
- Then he states principles of physics without reference to numbers, only using the physical ontology and ideology introduced above.
- The relation between the resulting theory and Newton's theory is a bit like the one between Euclid's 'synthetic' geometry and contemporary 'analytic' geometry.

⇒ Field's 'synthetic mechanics'

- The 'size' of Field's spacetime, which is homomorphic to $[R]^4$, is that of the powerset of the continuum.

⇒ Synthetic mechanics can 'simulate' arithmetic and real analysis in spacetime—even a version of the continuum hypothesis!

Question:

Did Field just replace real numbers with spacetime points/regions?

The conservativeness of mathematics

- Second step of Field's programme: show how mathematics can be added to the synthetic theories, and then establish that mathematics is conservative over the synthetic theories.

Definition (Conservativeness)

Let Φ be a sentence in the nominalistic language of a science N. Then the mathematical theory S is conservative over the scientific theory N if Φ is not a consequence of $S + N$ unless Φ is a consequence of N alone.

$\Rightarrow$ If mathematics were shown to be conservative over nominalistic science, then this would make mathematics dispensable.

$\Rightarrow$ Field aims to show that mathematics is conservative over science in terms of the nature of the subject matter of mathematics (particularly: its abstract ontology).

Question:

Where to draw the boundary between the abstract and the physical?

Analogy: Hilbert's programme and Field's programme

TABLE 9.1. The Hilbert and Field programmes compared

	Hilbert programme	Field programme
Basis:	finitary mathematics	nominalistic science
Instrument:	ideal mathematics	mathematics
Needed:	consistency	conservativeness

Figure: From Shapiro 2000, 235.

Incompleteness of Field's synthetic theory?

Shapiro 1983 argues that a problem analogous to Gödel's incompleteness theorem befalls Field's programme (by constructing a sentence G which is true of spacetime but is not derivable in the synthetic theory alone.

Stewart Shapiro (1983). Conservativeness and incompleteness. *Journal of Philosophy* 80: 521-531.

(2) Chihara's modal construction

- Charles Chihara (1932-2020)
- US American philosopher of mathematics and of logic
- Professor Emeritus at UC Berkeley
- *Constructibility and Mathematical Existence* (1990)

Model theory

- Model theory is an attempt to understand logical possibility and logical consequence in terms of a realm of set-theoretic constructions: to say that a given sentenece is logically possible is to say that there is a model that satisfies it.
- This permits replacing modal talk with talk about abstract objects such as sets and numbers (e.g. in Putnam)
- There is a group of philosophers of mathematics, to which belong the young Putnam and Chihara, who reverse this order: they deny the existence of mathematical objects and accept at least some form of modality as prior.

Sets first vs properties first

- Russell: replace references to sets with reference to properties or attributes (e.g. instead of talking of dogs, talk of the property of 'being a dog')
- Quine 1941: we need a criterion for when two properties are the same or distinct; slogan: no entity without identity
- Example: is the property of 'being an equilateral triangle' identical to the property of 'being an equiangular triangle'?
- Quine: attributes should give way to sets
- (But how does this solve the problem if the set of equilateral triangles is identical to the set of equiangular triangles?)

Chihara's account

- linguistic items instead of attributes

$\Rightarrow$ replace talk of sets with talk of open sentences (such as 'x is a dog')

- Chihara then turns to modality to obtain enough open sentences.
- He follows the Quinean slogan 'to be is to be the value of a bound variable', but introduces 'constructibility quantifier', which syntactically behaves like an existential quantifier:
- If Φ is a formula, x a certain type of variable, then $(Cx)\Phi$ is a formula, to be read as 'it is possible to construct an x such that Φ'.
- But: constructibility quantifiers carry no ontological commitment.

Many-sorted variables

(Many-sorted) variables on...

- Level 0: range over ordinary objects
- Level 1: range over open sentences satisfied by ordinary objects; the variables at this level can be bound by constructibility quantifiers
- Level 2: range over open sentences satisfied by level-1 open sentences
- etc

Assessing Chihara's approach

- Chihara doesn't intend to revise mathematics, but just to have the bulk of it come out true on an ontologically austere reading.
- His system is similar to ordinary type theory. He shows how to render any sentence of type theory into his system:
  - replace variables over sets of type n with level-n variables over open sentences;
  - replace membership (or predication) with satisfaction;
  - replace quantifiers over variables of level 1 with constructibility quantifiers.

⇒ His system is formally equivalent to that of simple type theory, but there is an important philosophical difference: it avoids ontological commitment—existence is replaced by constructibility (of open sentences).

(3) The critique of Burgess and Rosen 1997

John Burgess and Gideon Rosen (1997). *A Subject With No Object: Strategies for Nominalistic Interpretation of Mathematics* Oxford: Oxford University Press.

- Burgess and Rosen: crucial in 'stereotypical nominalist' arguments against realism based on epistemic difficulties of realism is the assumption of something like a causal theory of knowledge.
- On the other hand, the 'stereotypical realist' is moved by indispensability arguments and naturalized epistemology.
- As Burgess and Rosen note, many contemporary philosophers agree that we are warranted in the belief in mathematical objects iff mathematics is indispensable for science.
- Burgess and Rosen: realists should not accept this framing of the debate

Burgess and Rosen's assessment of nominalism

- Burgess and Rosen state that, in Quinean terms, nominalists trade ontology for ideology.
- Question: what is an austere nominalistic theory to be be used for? Burgess and Rosen see two main options:

1. revolutionary approach: replace classical theory; they identify two subtypes:
    1. first-philosophy orientation, i.e., the motivation is philosophical
    2. naturalistic orientation
2. hermeneutic approach: reconstructed nominalistic theory provides the underlying meaning of the original scientific theory

As far as nominalists like Field are concerned, this is arguably a false dilemma!

(4) Addendum: Young Turks

- Shapiro ends the chapter with a discussion of two important contributions to the debate of the 1990s, which claim that the issue must be transcended as the philosophical dispute cannot be decided. This leads to the

Question:

What is it about the practice of mathematics and science that allows them to proceed with terms that refer to objects with which we have no causal contact?

(A) Jody Azzouni

Jody Azzouni (1994). *Metaphysical Myths, Mathematical Practice*. Cambridge: Cambridge University Press.

- The book is centrally concerned with the nature of reference and truth in mathematics and defends the following

Thesis

It is mathematical practice which fixes mathematical reference if anything does, and so the philosopher needs to pay attention to practice.

(B) Mark Balaguer

Mark Balaguer (1998). *Platonism and Anti-Platonism in Mathematics*. Oxford: Oxford University Press.

1. There is exactly one tenable version of Platonism and it is immune from any rational challenges.
2. There is exactly one tenable version of anti-Platonism, and this view is likewise invincible.
3. The epistemic dilemma is due to there being no fact of the matter as to whether mathematical objects exist.