The mathematical formalism and the standard way of thinking about it

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MA Seminar: Philosophy of Physics

Vectors and vector spaces

Albert, *Quantum Mechanics and Experience*, Ch. 2

> *"Vectors, in quantum mechanics, are going to represent physical states of affairs. The addition of vectors will turn out to have something to do with the superposition of physical states of affairs." (20)*

Definition (Vector space)

A vector space V over a field K (typically, $K = \mathbb{R}$ or $K = \mathbb{C}$) is a set of elements, called vectors, in which two operations, addition and multiplication by an element of K (called a scalar), are defined by the following axioms:

(i) $|A\rangle + |B\rangle = |B\rangle + |A\rangle$,

(ii) $(|A\rangle + |B\rangle) + |C\rangle = |A\rangle + (|B\rangle + |C\rangle)$,

(iii) *There exists a zero vector* $|0\rangle$ *s.t.* $|A\rangle + |0\rangle = |A\rangle$,

(iv) *For any* $|A\rangle$, *there exists* $-|A\rangle$, *s.t.* $|A\rangle + (-|A\rangle) = |0\rangle$,

(v) $c(|A\rangle + |B\rangle) = c|A\rangle + c|B\rangle$,

(vi) $(c + d)|A\rangle = c|A\rangle + d|A\rangle$,

(vii) $(cd)|A\rangle = c(d|A\rangle)$,

(viii) $1|A\rangle = |A\rangle$,

where $|A\rangle, |B\rangle, |C\rangle \in V$ *and* $c, d \in K$ *and 1 is the unit element of* K.

Scalar product (or inner product)

Definition (Scalar or inner product)

The scalar or inner product $\langle A|B\rangle$ *of two vectors is defined as*

$$\langle A|B\rangle = |A||B|\cos\theta,$$

(a number) where θ *is the angle between the two vectors and* $|A|$*, the length or norm of* $|A\rangle$*, is the square root of the number* $\langle A|A\rangle$*.*

- vectors plus vectors are vectors
- vectors times numbers are vectors
- vectors times vectors are numbers

Definition (Orthogonality)

If $|A| \neq 0$ *and* $|B| \neq 0$ *and yet* $\langle A|B\rangle = 0$*, then* $|A\rangle$ *and* $|B\rangle$ *are orthogonal to one another.*

(Fact: $\cos 90^\circ = 0$)

Dimensionality and bases

Definition (Dimensionality of vector spaces)

The dimension of a vector space is equal to the maximum number N of vectors $|A_1\rangle, |A_2\rangle, ..., |A_N\rangle$ which can be chosen in the space such that for all values of i and j from 1 through N such that $i \neq j$, $\langle A_i|A_j\rangle = 0$. The dimension of a space is equal to the number of pairwise orthogonal vectors. (cf. Albert, 21)

- For all vector spaces with dim ≥ 2, there are infinitely many such sets of pairwise orthogonal vectors.
- If all the vectors in such a set have a unit norm, they are said to form an orthonormal basis of that vector space.
- Suppose that the set $|A_1\rangle, ..., |A_N\rangle$ forms a basis of an N-dimensional vector space V. Then any vector $|B\rangle \in V$ can be expressed as

$$|B\rangle = b_1|A_1\rangle + \cdots + b_N|A_N\rangle,$$

where the expansion coefficients b_i are the numbers $b_i = \langle B|A_i\rangle$.

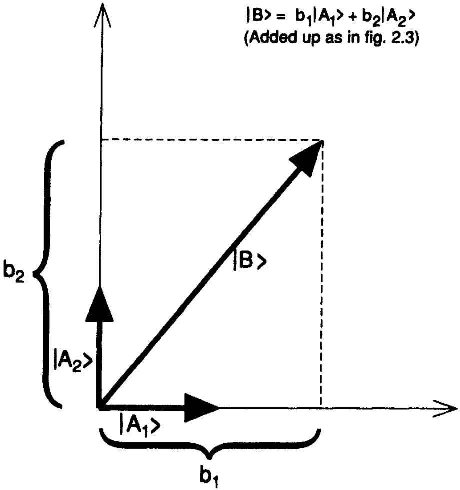


Figure: Figure 2.5 in Albert (1992)

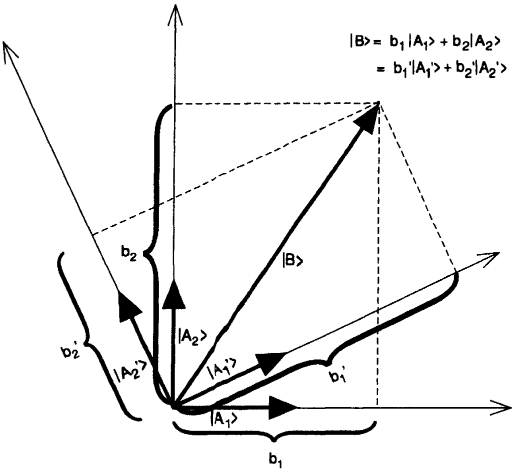


Figure: Figure 2.6 in Albert (1992)

Since

$$\langle A|(|B\rangle + |C\rangle) = \langle A|B\rangle + \langle A|C\rangle,$$

we find

$$|X\rangle + |Y\rangle = (x_1 + y_1)|A_1\rangle + \cdots + (x_N + y_N)|A_N\rangle \quad (1)$$

and

$$\langle X|Y\rangle = x_1 y_1 + \cdots + x_N y_N, \quad (2)$$

where the x_i and y_i are the expansion coefficients of $|X\rangle$ and $|Y\rangle$, respectively, in a given basis $|A_i\rangle$.

Important: while the coefficients depend on the chosen basis, the sum $\langle X|Y\rangle$ does not; i.e., the scalar product is invariant under changes of basis.

For a given basis, an N-dim vector with coefficients x_i in this basis can be represented as a column of N numbers:

$$|X\rangle = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}. \tag{3}$$

It follows from (2) that the norm $|X|$ of a vector $|X\rangle$ is given by

$$|X| = \sqrt{(x_1)^2 + \cdots + (x_N)^2},$$

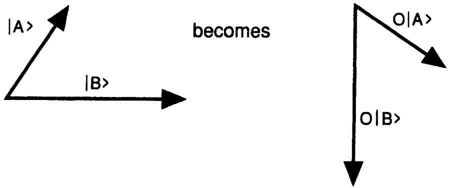
which is also invariant under changes of basis.

Operators

Definition (Operator)

An operator $\hat{O}$ defined on a vector space V is a map $\hat{O}: V \rightarrow V, |A\rangle \mapsto |A'\rangle = \hat{O}|A\rangle$ where $|A\rangle, |A'\rangle \in V$, i.e., a prescription for taking every vector in V into some other vector in V.

Some examples: unit operator $\hat{1}$ transforming every vector into itself; 'multiply every vector by the number 7'; 'rotate every vector clockwise by 90° about some particular $|C\rangle$' (depicted below for a $|C\rangle$ pointing out of the page); 'map every vector into some particular $|A\rangle$'; etc.



Definition (Linear operators)

A linear operator on a vector space V is defined as an operator with the properties

$$\hat{O}(|A\rangle + |B\rangle) = \hat{O}|A\rangle + \hat{O}|B\rangle \tag{4}$$

and

$$\hat{O}(c|A\rangle) = c(\hat{O}|A\rangle), \tag{5}$$

where $|A\rangle, |B\rangle \in V$ and any number c.

- Question: which of the examples of operators on previous page are linear?
- Linear operators on an N-dim vector space can be represented by arrays of N^2 numbers called matrices (singular: matrix). Example for an operator defined on a 2-dim vector space:

$$\hat{O} = \left[\begin{array}{cc} O_{11} & O_{12} \\ O_{21} & O_{22} \end{array} \right]. \tag{6}$$

The matrix coefficients O_{ij}—numbers—are defined as

$$O_{ij} = \langle A_i | \hat{O} | A_j \rangle$$

in a basis $|A_1\rangle, ..., |A_N\rangle$.

Rule for multiplying operator matrices with vector columns, exemplified in 2-dimensional case:

$$\begin{aligned} \hat{O}|B\rangle &= \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} (O_{11}b_1 + O_{12}b_2) \\ (O_{21}b_1 + O_{22}b_2) \end{bmatrix} \\ &= (O_{11}b_1 + O_{12}b_2)|A_1\rangle + (O_{21}b_1 + O_{22}b_2)|A_2\rangle. \end{aligned}$$

Eigenvectors and eigenvalues

Definition (Eigenvectors and eigenvalues)

In case that for some particular operator $\hat{O}$ and some particular vector $|X\rangle$

$$\hat{O}|X\rangle = x|X\rangle$$

for some number x, then $|X\rangle$ is an eigenvector of $\hat{O}$, with eigenvalue x.

- certain vectors will in general be eigenvectors of some operators and not of others
- certain operators will in general have some vectors, and not others, as eigenvectors
- other operators will have other vectors as eigenvectors.
- The operator-eigenvector relation depends only on the vector and the operator in question, but not on the basis chosen.
- Look at the examples in the textbook.

Quantum mechanics: the five basic principles

Principle (A: Physical states)

Every physical system, composite or simple, is associated with some particular vector space. Every unit vector in this space (the 'state vectors') represents a possible physical state of the system. The states picked out by these vectors are taken to comprise all of the physically possible situations, although the correspondence is not one-to-one.

Principle (B: Observables)

Measurable properties of physical systems ('observables') are represented by linear operators on the vector spaces associated with those systems. The rule connecting the operators and the vectors states that if the vector happens to be an eigenvector (with eigenvalue, say, a) of an operator in question, then the state corresponding to the vector has the value a of that particular measurable property associated with the operator.

Back to the hardness and color example

Construct a vector space in which the states of being hard and being soft can be represented:

$$|\text{hard}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\text{soft}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{7}$$

which implies, by (2), $\langle\text{hard}|\text{soft}\rangle = 1 \cdot 0 + 0 \cdot 1 = 0$. In fact the two vectors in (7) form a basis of the vector space. Which operator should represent the observable ‘hardness’?

$$\text{hardness operator} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{8}$$

where it is stipulated that eigenvalue $+1$ means ‘hard’ and -1 ‘soft’ (vectors in (7) are eigenvectors of operator in (8)).

'Black' and 'white' states 'superpositions' of both the 'hard' and the 'soft' states, which means, since superposition states are states in the same vector space, that 'black' and 'white' ought to be representable in the same vector space. This plays out as follows:

$$
\begin{aligned}
|\text{black}\rangle &= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad |\text{white}\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \\
\text{color operator} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\end{aligned}
\tag{9}
$$

where 'color $= +1$' means 'black' and 'color $= -1$' means 'white'. Also, $\langle\text{black}|\text{white}\rangle = 0$, and $|\text{black}\rangle$ and $|\text{white}\rangle$ constitute another basis of the same space.

Recall (essentially from (1)) that if $|A\rangle = (a_1, a_2)$ and $|B\rangle = (b_1, b_2)$ (new representations of column vectors), then

$$|A\rangle + |B\rangle = \begin{bmatrix} (a_1 + b_1) \\ (a_2 + b_2) \end{bmatrix}.$$

One can then see from (7), (8), and (9) how superposition and incompatibility is encoded in formalism:

$$\begin{aligned} |\text{black}\rangle &= \frac{1}{\sqrt{2}}|\text{hard}\rangle + \frac{1}{\sqrt{2}}|\text{soft}\rangle, \\ |\text{white}\rangle &= \frac{1}{\sqrt{2}}|\text{hard}\rangle - \frac{1}{\sqrt{2}}|\text{soft}\rangle, \\ |\text{hard}\rangle &= \frac{1}{\sqrt{2}}|\text{black}\rangle + \frac{1}{\sqrt{2}}|\text{white}\rangle, \\ |\text{soft}\rangle &= \frac{1}{\sqrt{2}}|\text{black}\rangle + \frac{1}{\sqrt{2}}|\text{white}\rangle. \end{aligned} \tag{10}$$

Exercise for the reader: convince yourself that $|\text{black}\rangle$ and $|\text{hard}\rangle$ are not eigenvectors of the color operator (and vice versa, mutatis mutandis).

The hardness and color operators are incompatible with one another, in the sense that states of definite hardness have no assignable color value, and vice versa.

Principle (C: Dynamics)

Given the state of any physical system at any 'initial' time, and given the forces and constraints to which the system is subject, the Schrödinger equation gives a prescription whereby the state of that system at any other time is uniquely determined. This dynamics of the state vector is thus deterministic.

The dynamical laws are linear: if any state $|A\rangle$ at t_1 is evolved into another state $|A'\rangle$ at t_2 and any $|B\rangle$ at t_1 is evolved into $|B'\rangle$ at t_2, then $\alpha|A\rangle + \beta|B\rangle$ at t_1 is evolved into $\alpha|A'\rangle + \beta|B'\rangle$.

We know what happens if we measure a state with respect to a particular property when that state is in an eigenstate of the operator corresponding the measurable property in question (what?). But what happens if it isn't?

Principle (D: Connection with experiment, 'Born's rule')

A measurement of the observable $\hat{A}$ is performed on a system in state $|b\rangle$, where the eigenvectors of $\hat{A}$ are $|a_i\rangle$ with eigenvalues a_i, i.e. $\hat{A}|a_i\rangle = a_i|a_i\rangle$ for all i. The probability that the outcome of such a measurement will be a_i is equal to

$$|\langle b|a_i\rangle|^2.$$

Remarks:

- The number $|\langle b|a_i\rangle|^2$ is in the interval $[0, 1]$.
- In the special case when the system is in an eigenstate of the operator corresponding to the measurement, we get with probability 1 that the outcome is the eigenvalue associated with the eigenstate.
- Probability that black electron is found to be hard is 1/2, as it should be.

Principle (E: Collapse)

Whatever the state vector of a system S was just prior to a measurement of an observable O, the state vector of S just after the measurement must be an eigenvector of O with an eigenvalue corresponding to the outcome of that measurement.

Remarks:

- Which eigenvector the system jump into is determined by the outcome of the measurement; and this outcome, by Principle D, is a matter of probability.

$\Rightarrow$ element of chance, indeterminism, enters into the evolution of the state vector

Homework: read carefully how all of this is implemented in Albert pp. 36-38

- Notice that Principle C was supposed to be a completely general account of how the state vector evolves under any circumstances, while Principle E seems to be a special case of C, but can't obviously be deduced it...
- Measurement in QM is (according to the standard view) a very active process that invariably changes the measured system
- Albert: "That's what's at the heart of the standard view. The rest... is details." (38)

Complex vector spaces

QM: complex vector spaces, i.e., vectors can be multiplied by complex numbers

- $\Rightarrow$ expansion coefficients also complex
- $\Rightarrow$ necessitates a few changes to assure that the norm of vectors is a positive real number, that probabilities are positive real numbers in $[0, 1]$

- $|A\rangle$ and $c|A\rangle$ represent the same physical state, where $c \in \mathbb{C}$ is any of the infinitely many complex numbers of absolute value 1

Definition (Hermitian operators)

A Hermitian operator is a linear operator such that all its associated eigenvectors only have real eigenvalues.

- Operators associated with measurable properties must be Hermitian.

Properties of Hermitian operators

1. If two vectors are both eigenvectors of the same Hermitian operator, and if the eigenvalues associated with these eigenvectors are two different numbers, then the two vectors in question are orthogonal to each other (otherwise measurements wouldn't be repeatable):

$$\hat{A}|a_1\rangle = a_1|a_1\rangle, \hat{A}|a_2\rangle = a_2|a_2\rangle, \text{with } \mathbb{R} \ni a_1 \neq a_2 \in \mathbb{R}$$
$$\Rightarrow \langle a_1|a_2\rangle = 0.$$

2. Any Hermitian operator on an N-dimensional space will always have at least one set of N mutually orthogonal eigenvectors.

$\Rightarrow$ It's always possible to form a basis of the space out of the eigenvectors of any Hermitian operator.

3. If a Hermitian operator on an N-dimensional space has N different eigenvalues, then there is a unique vector in the space associated with each one of these eigenvalues. These operators are called complete or nondegenerate operators.
4. Any Hermitian operator on a given space will invariably be associated with some measurable property of the physical system connected to that space.
5. Any vector in a given space is an eigenvector of some complete Hermitian operator on that space.

$\Rightarrow$ Every quantum-mechanical system has an infinity of mutually incompatible measurable properties.

Definition (Commutator)

The commutator of two matrices A and B, denoted by $[A, B]$, is defined to be the object $AB - BA$.

- If $[A, B] = 0$, then A and B share at least one set of eigenvectors which form a basis of the space.
- In other words, the commutators of incompatible observable matrices are nonzero.

$\Rightarrow$ property of commutativity is convenient mathematical test for compatibility

Application: coordinate space

- Principle of correspondence: requirement that 'classical limit' is Newtonian mechanics
- Albert: requirement can be "parlayed" into a prescription of constructing a quantum theory from a classical theory such that the quantum operators correspond to measurable properties of classical theory

$\Rightarrow$ momentum and position of a particle are incompatible: $[\hat{X}, \hat{P}] = i\hbar$

- basis of position eigenvectors: $|x\rangle$, position operator $\hat{X}$
- Note: possible eigenvalues form a continuum from $-\infty$ to $+\infty$
- eigenvectors of $\hat{X}$ form a basis of the state space of the particle, but there are infinitely many different eigenvalues, and thus eigenvectors, and these eigenvectors are orthogonal

$\Rightarrow$ state space of the particle is ∞-dimensional

- Any vector $|\Psi\rangle$ in that ∞-dimensional space can be expressed as expansion in terms of eigenstates $|x\rangle$, with expansion coefficients as function of x

$$a_x = \langle\Psi|x\rangle = \Psi(x)$$

  Just as the vector in (3) is fully determined by the N numbers in the column, in a given basis, the function $\Psi(x)$ (an infinite list) determines the unique vector $|\Psi\rangle$ in the ∞-dimensional space (in the $\hat{X}$ basis implicit here).

- Probabilities can also be read off the expansion coefficients: if wave function is $\Psi(x)$ and a position measurement is performed, then the probability that the particle will be found at position $x = x_1$ is given by $|\Psi(x_1)|^2$

Application: bipartite systems

- state of bipartite system of particle 1 in state $|\Psi_a\rangle$ and particle 2 in state $|\Psi_b\rangle$ given by $|\Psi_a\rangle_1|\Psi_b\rangle_2$ or by $|^1\Psi_a,^2\Psi_b\rangle$
- If the two particles don't interact the joint probability that the outcome of measurement of $\hat{A}$ on particle 1 is a and that the outcome of measurement of $\hat{B}$ on particle 2 is b is the probability of the former outcome times the probability of the latter outcome
- Rule for multiplying vectors representing bipartite states:

$$\langle^1\Psi_i,^2\Psi_j|^1\Psi_k,^2\Psi_l\rangle = \langle^1\Psi_i|^1\Psi_k\rangle \cdot \langle^2\Psi_j|^2\Psi_l\rangle$$

  ($= 0$ unless $i = k$ and j=l).
- dimensionality of two-particle state space is the product of the dimensionalities of the state spaces for the individual particles (e.g. N^2 if both single-particle state spaces have dimension N, with N^2 orthogonal vectors $|^1\Psi_i,^2\Psi_j\rangle$ as basis for bipartite state space)

Non-separable state

$$|Q\rangle = \frac{1}{\sqrt{2}}|^1\Psi_1,^2\Psi_1\rangle + \frac{1}{\sqrt{2}}|^1\Psi_2,^2\Psi_2\rangle \tag{11}$$

is a non-separable or entangled state because it cannot be recast in the form $|^1 f,^2 g\rangle$, i.e., it cannot be decomposed into a well-defined state of particle 1 and a well-defined state of particle 2.

Example:

$$|Q'\rangle = \frac{1}{\sqrt{2}}|^1\hat{X} = 5,^2\hat{X} = 7\rangle + \frac{1}{\sqrt{2}}|^1\hat{X} = 9,^2\hat{X} = 11\rangle$$

Anything about either of the particles has any definite value here, but the difference in their position does:

$$(^2\hat{X} -^1\hat{X})|Q'\rangle = 2|Q'\rangle.$$

Homework: study how Principles D and E pan out for the bipartite system in general

Examples for collapsing bipartite systems
Separable states

Separable states: measurement only affects state of measured particle; example: state prior to measurement is

$$|S\rangle = |^1\hat{M} = m, ^2\hat{N} = n\rangle$$

and then $^1\hat{A}$ is measured, with outcome a_7. What is the post-measurement state? Expand $|S\rangle$ in the basis of eigenstates of $^1\hat{A}$ and $^2\hat{N}$ and delete all the terms with $^2\hat{N} \neq n$:

$$|S\rangle = s_1|^1\hat{A} = a_1, ^2\hat{N} = n\rangle + s_2|^1\hat{A} = a_2, ^2\hat{N} = n\rangle + \dots$$

with

$$s_i = \langle ^1\hat{A} = a_i, ^2\hat{N} = n|^1\hat{M} = m, ^2\hat{N} = n\rangle = \langle ^1\hat{A} = a_i|^1\hat{M} = m\rangle.$$

Post-measurement state:

$$|S'\rangle = |^1\hat{A} = a_7, ^2\hat{N} = n\rangle.$$

Non-separable states

Non-separable states: measurement also brings about changes in the description of the unmeasured particle; example: pre-measurement state is

$$|NS\rangle = \frac{1}{\sqrt{2}}|^1\hat{A} = a_4, ^2\hat{K} = k_9\rangle + \frac{1}{\sqrt{2}}|^1\hat{A} = a_5, ^2\hat{K} = k_{17}\rangle$$

where, again, $^1\hat{A}$ is measured, yielding the result a_5. The formula above is already written down in terms of eigenstates of $^1\hat{A}$ and $^2\hat{K}$, so can directly proceed to delete all the terms other than the ones for which $^1\hat{A} = a_5$. We obtain:

$$|NS'\rangle = |^1\hat{A} = a_5, ^2\hat{K} = k_{17}\rangle.$$

Degrees of freedom

- Bipartite systems are important because any system with multiple degrees of freedom is essentially described analogously in QM.
- degrees of freedom (DOFs): moving in one spatial dimension, moving in another spatial dimension, the having of color-hardnes properties, etc
- Quantum Field Theory (QFT): Principles A-D still valid, but particle ontology replaced by field ontology, which conceives of world as consisting of infinite arrays of infinitely small physical systems at every point of space, interacting with one another in particular ways

$\Rightarrow$ infinitely many DOFs

Application: Two-path experiment

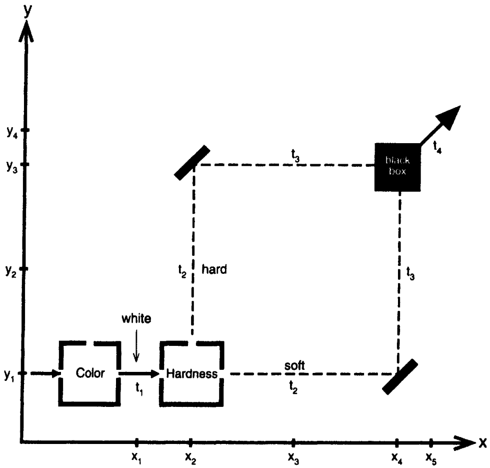


Figure: The two-path experiment reconsidered (Albert, Fig. 2.8)

At t_1, before entering hardness box, the state of the electron is

$$\begin{aligned}
|\text{white}, \hat{X} = x_1, \hat{Y} = y_1\rangle &= \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} |\hat{X} = x_1, \hat{Y} = y_1\rangle \\
&= (\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}) |\hat{X} = x_1, \hat{Y} = y_1\rangle \\
&= \frac{1}{\sqrt{2}} |\text{hard}, \hat{X} = x_1, \hat{Y} = y_1\rangle \\
&\quad - \frac{1}{\sqrt{2}} |\text{soft}, \hat{X} = x_1, \hat{Y} = y_1\rangle \\
&= \frac{1}{\sqrt{2}} |a\rangle - \frac{1}{\sqrt{2}} |b\rangle.
\end{aligned}$$

What if the state at t_1 had only been one of these (normalized) terms (and the hardness box is still a hardness box)?

If at t_1 it had just been $|a\rangle$, then, at t_2, it would simply have been $|\text{hard}, \hat{X} = x_2, \hat{Y} = y_2\rangle$. If it had been $|b\rangle$ at t_1, then it would have been $|\text{soft}, \hat{X} = x_3, \hat{Y} = y_1\rangle$ at t_2.

It follows from this and from the linearity of the dynamics (cf. Principle D) that the state at t_2 is really

$$\frac{1}{\sqrt{2}}|\text{hard}, \hat{X} = x_2, \hat{Y} = y_2\rangle - \frac{1}{\sqrt{2}}|\text{soft}, \hat{X} = x_3, \hat{Y} = y_1\rangle$$

But now we can no longer factor out the position and the spin properties part of the eigenstate, which means it's non-separable, i.e. there are non-separable correlations between spin and coordinate-space properties of the e^-. This means that neither any spin property, nor any coordinate-space property of the e^- has any definite value here. The only definite properties at t_2 involve combinations of spin-space and coordinate-space variables of the particle. According to the standard view, it just doesn't make sense to talk of any spin or coordinate-space properties of the e^- at t_2.

Similarly, the state of the e$^-$ at t_3 is

$$\frac{1}{\sqrt{2}}|\text{hard}, \hat{X} = x_3, \hat{Y} = y_3\rangle - \frac{1}{\sqrt{2}}|\text{soft}, \hat{X} = x_4, \hat{Y} = y_2\rangle$$

and at t_4, it is

$$\begin{aligned} \frac{1}{\sqrt{2}}|\text{hard}, \quad \hat{X} = x_5, \quad \hat{Y} = y_4\rangle - \frac{1}{\sqrt{2}}|\text{soft}, \hat{X} = x_5, \hat{Y} = y_4\rangle \\ = \quad \frac{1}{\sqrt{2}}(|\text{hard}\rangle - |\text{soft}\rangle)|\hat{X} = x_5, \hat{Y} = y_4\rangle \\ = \quad |\text{white}, \hat{X} = x_5, \hat{Y} = y_4\rangle. \end{aligned}$$

This state is separable again, has definite position and spin/color properties.

Now suppose that the experiment were stopped at t_3 by measuring the electron's position. Then a collapse would occur, and the state just after the measurement would be

$$|\text{hard}, \hat{X} = x_3, \hat{Y} = y_3\rangle \text{ or } |\text{soft}, \hat{X} = x_4, \hat{Y} = y_2\rangle$$

each with probability 0.5. If the measurement were performed at t_4, then the state would be

$$|\text{hard}, \hat{X} = x_5, \hat{Y} = y_4\rangle \text{ or } |\text{soft}, \hat{X} = x_5, \hat{Y} = y_4\rangle.$$

What if we were to put a wall into the soft path at (x_3, y_1)? Then the state at t_4 would be

$$\frac{1}{\sqrt{2}}|\text{hard}, \hat{X} = x_5, \hat{Y} = y_4\rangle - \frac{1}{\sqrt{2}}|\text{soft}, \hat{X} = x_3, \hat{Y} = y_1\rangle.$$

- In other words, the state would remain entangled between spin and coordinate-space properties even at t_4.
- If a position measurement would then be performed at t_4, the e^- would be found at (x_5, y_4) with probability 0.5 (in which case it would be hard), or at (x_3, y_1) with probability 0.5 (in which case it's soft).
- But it's a QM-state, and thus a vector in the state space of this e^-. In virtue of fact (5) about Hermitian operators, then, it must be an eigenstate of some operator, i.e. it must be associated with definite values of some measurable properties.
- To bring out part of what these properties might be, study Albert's introduction of the operators 'where' and 'zap' (pp. 57f)