# Logicism

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**Introduction to Philosophy of Mathematics**

# Logicism

Bertrand Russell (1919). *Introduction to Mathematical Philosophy*. London: Allen and Unwin.

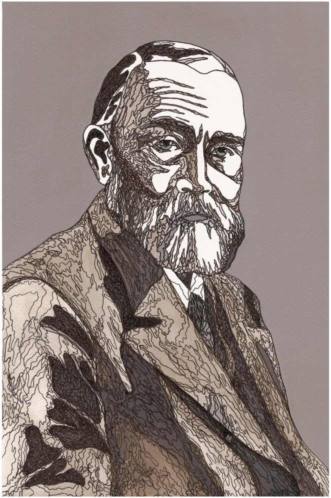
## Bertrand Russell (1919, Chapter 18)

*Mathematics and logic, historically speaking, have been entirely distinct studies... But both have developed in modern times: logic has become more mathematical and mathematics has become more logical. The consequence is that it has now become wholly impossible to draw a line between the two; in fact the two are one... The proof of their identity is, of course, a matter of detail.*

**Characterization (Logicism)**

*Logicism is the thesis that at least parts of mathematics, notably arithmetic and geometry, are reducible to logic, possibly augmented by set theory.*

# (Friedrich Ludwig) Gottlob Frege (1848-1925)



- German mathematician, logician, philosopher
- studies of mathematics, physics, and philosophy at Jena and Göttingen (PhD 1873)
- one of the founding fathers of modern logic and analytic philosophy (philosophy of language, philosophy of math)
- *Begriffsschrift* (1879), *Grundlagen der Arithmetik* (1884), *Grundgesetze der Arithmetik* (1893, 1903)
- mostly ignored during his lifetime, but notable influence on Peano, Russell, Wittgenstein, Carnap

# Analyticity and a priority as epistemic concepts

## Characterization (Analyticity and a priority as epistemic concepts)

*"The problem becomes... that of finding the proof of the proposition, and of following it up right back to the primitive truthts. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one... If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some general science, then the proposition is a synthetic one. For a truth to be a posteriori, it must be impossible to construct a proof of it without including an appeal to facts, i.e., to truths which cannot be proved and are not general... But if, on the contrary, its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof, then the truth is a priori." (Frege 1884, §3)*

# Frege and the foundations of mathematics

- Frege is a double realist (so logic has ontology, will be needed to underwrite the existence of natural numbers), held arithmetic propositions to be analytic

$\Rightarrow$ need to derive them from general logical laws and definitions

- and that's just the logicist program
- Frege defined equinumerosity without appeal to numbers (one-one correspondence), but only (second-order) logic, by proposing 'Hume's principle':

**Principle (Hume)**

*For any concepts $F$, $G$, the number of $F$ is identical to the number of $G$ iff $F$ and $G$ are equinumerous.*

You have already seen most of what follows, but with 'concept', 'extension', 'number' etc, replaced by 'property', 'set'/'class', 'cardinality'.

# Frege's theorem

**Frege's theorem**

Given Hume's principle, the basic principles of arithmetic hold.

- number zero defined as the 'number of the concept' of not being identical to itself
- successor relation: essentially, $n$ is a successor to $m$ just in case "there is a concept which applies to exactly $n$ objects and when we remove one of those objects, $m$ objects remain." (110)
- concept 'identical to zero' holds of exactly one object—the number zero—, using the successor relation, the number one can now be defined to be the number of the concept 'identical to zero'
- next step: define number two as the number of the concept 'either identical to zero or identical to one', etc.

$\Rightarrow$ $\exists$ infinitely many natural numbers

# Defining natural numbers and the Caesar problem

- It is circular to say that $n$ is a natural number just in case it can be constructed as above in finitely many steps, so:

**Definition (Natural number)**

*"n is a natural number if n falls under every concept which holds of zero and is closed under the successor relation." (111) (i.e., it falls under every concept which holds of zero and if any object falls under it, then all of the object's successor fall under it as well)*

- Hume's principle legislates identities such as 'the number of $F$ = the number of $G$', but not of identies identifying any of these numbers with a singular term (such as the numeral '2'), i.e., it doesn't determine whether 'the number of my parents = 2' or 'the number of my parents = Julius Caesar', but only 'the number of my parents = the number of your parents'
- That's the Caesar problem, i.e., the problem of identifying the natural numbers (as objects).

# Grounding Hume's principle

**Definition (Extension)**

*"The extension of a concept is the class of all objects that concept applies to." (112)*

This allows Frege to define natural numbers via concepts and their extensions:

**Definition (Natural numbers, independent of Hume's principle)**

*"The number which belongs to the concept $F$ is the extension of the concept equinumerous with the concept $F$." (ibid.)*

- number two is class of all concepts which obtain of exactly two objects (e.g., concept of being my parent is an element of the number two)
- Frege: these definitions (together with some common properties of extensions) suffice to derive Hume's principle

# Frege's building falls apart...

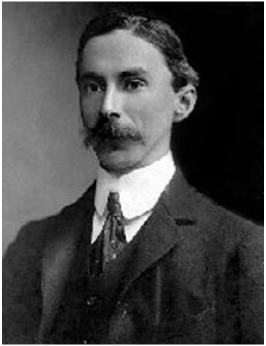
- *Grundgesetze* (1893, 1903): fuller development of this derivation based on the infamous 'Basic Law V', which asserts that, intuitively enough

## Basic Law V

"For any concepts $F$, $G$, the extension of $F$ is identical to the extension of $G$ if and only if for every object $a$, $Fa$ if and only if $Ga$." (114)

- In June 1902, Russell sent a letter to Frege, showing how Basic Law V is inconsistent—it leads to Russell's paradox! (for details, cf. 114f)
- After some failed attempts, Frege abandons the logicist project.

# Sir Bertrand Russell (1872-1970)



- graduated from Trinity College, Cambridge
- *Principia Mathematica* (1910-13), *The Problems of Philosophy* (1912)
- prolific writer, Nobel Prize in Literature in 1950
- one of the founders of modern analytic philosophy
- important contributions to logic, philosophy of mathematics, philosophy of language, epistemology, philosophy of science, ethics, philosophy of religion, political philosophy
- social activist; opposed British involvement in WW1, Soviet Union under Lenin and Stalin, nuclear arms, and American involvement in the Viet Nam war; jailed twice (2nd time at age 89!)

# Russell's resolution

- Russell believed that logicism can be saved from inconsistency.
- He considered impredicative definitions circular and hence the source of the inconsistency.

$\Rightarrow$ circular impredicative definitions illegitimate ('vicious circle principle')

- definition of $R$ is impredicative (contradiction is derived from assumption that definition of $R$ holds of its own extension)
- development of type theory (cf. handout on set theory)
- put concepts aside, talk of 'classes' and 'extensions'
- in reconstruction of arithmetic only allows predicative definition (unlike Frege)

# Russellian numbers

**Definition**

*"For any class $C$, define the number of $C$ to be the 'class of all those classes that are' equinumerous with $C$." (117)*

- $A$: class of my four grandparents, i.e. $A$ is of type 1 (grandparents are objects which are not classes)
- number of $A$ is the class of all four-element type-1 classes, and so is of type 2
- number of a type-$N$ class is a type-$(N+1)$ class
- zero: type-2 class with exactly one element, the type-1 empty set
- one: type-2 class of all type-1 classes with a single element, i.e., it has as many elements are there are individuals of type 0
- etc.; number of class $A$ is four
- In fact, we can introduce different natural numbers for each type.

- translated to classes, Frege's 1 would be defined as the number of the class whose only element is the number 0, i.e., the number of $\{0\}$
- but: $\{0\}$ is not a member of Russell's number 1, but a type-3 class (because 0 is type 2), so its number would be type 4
- Frege's 2 would be the number of the class $\{0, 1\}$—but which 1?
    1. It can't be Russell's 1 because then the class $\{0, 1\}$ would contain a pair of type-2 classes and hence be a type-3 class, so the Frege-2 (its number) would have to be type 4, but we wanted to define a number 2 of type 2.
    2. It can't be the 1 defined as the type-3 class consisting of all type-2 classes with a single element because the class consisting of 0 and this 1 doesn't exist, since it contains a type-2 class ('0') and a type-3 class ('1' defined as here).
- Frege's proof that $\exists$ infinitely many natural numbers doesn't go through anymore in Russell's type theory.

# Introducing new axioms

- Russell: each natural number must be underwritten by that many individuals in universe

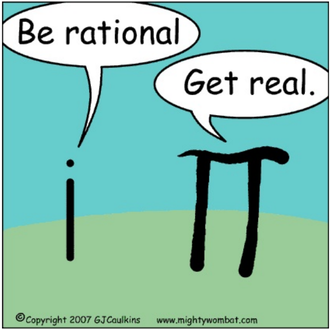
$\Rightarrow$ Russell and Whitehead introduce axiom of infinity, asserting the existence of infinitely many individuals

- But this axiom cannot be proved, it is not analytic or a priori or true by necessity.

$\Rightarrow$ 'conditional logicism': if there exist infinitely many individuals, then such-and-such proposition in arithmetic

- disallowing impredicative definitions also necessitated the introduction of another axiom: the principle of reducibility, stating that "at each type, for every class $c$, there is a predicative (level 0) class $c'$ which has the same members of $c$" (120)
- assuming infinity and reducibility, Russell and Whitehead could derive the standard Peano axioms for arithmetic

## Getting more numbers




- integer and rational numbers can be obtained as relations on natural numbers
- real numbers introduced using the concept of 'Dedekind cuts' such that real numbers are classes of rational numbers (real analysis also requires the axiom of choice)
- complex numbers are ordered pairs of real numbers

**Shapiro (2000, 123)**

*With the principles of infinity, reducibility, and choice, Whitehead and Russell's type theory captures just about every branch of pure mathematics short of set theory.*

# Rudolf Carnap (1891-1970) and logical empiricism

- studied physics and logic at Jena and Berlin
- 1926 appointed at U Vienna, 1931 U Prague, 1936 U Chicago, 1954 UCLA
- *Der logische Aufbau der Welt* (1928), *Meaning and Necessity* (1956)
- one of the leaders of logical positivism

# From Carnap's FBI file...

http://vault.fbi.gov/Rudolph%20Carnap/

(Part 2, page 3)

> *Professor [...] CARNAP, also French, now at the University of Chicago, is reported to have influenced Neo-Thomism and other French Catholic philosophical doctrines toward Logical Empiricism.*

# Logical empiricism and the semantic tradition

- logical positivism/empiricism: mathematics is analytic and a priori (contrast to Mill)
- source of necessity/a priority in language: necessary truth is truth by definition, a priori knowledge is knowledge of language use
- Existence of numbers: 'there are prime numbers greater than 10' implies 'there are numbers'—should we thus believe in existence of numbers
- Carnap: yes and no, but statements about objective existence of numbers 'meaningless'
- idea of linguistic framework: language about kind of entities

$\Rightarrow$ internal vs. external questions

- internal questions: questions of existence of entities within the framework
- external questions: existence or reality of the system of entities as whole

# The number framework

- internal questions (such as ‘Is there a prime number greater than 100?’) are analytical, logically true
- external question regarding the existence of numbers is meaningless
- whether framework should be adopted is purely pragmatic issue
- realists (such as Gödel): impredicative definitions are ok because numbers, classes etc have independent existence
- logical empiricists (such as Carnap): ok for purely pragmatic reasons
- mathematical truths are a priori because they lack factual content
- logical empiricists: there are no synthetic a priori truths

# An objection

> *Every [mathematical] proposition p is associated with its framework P. Knowledge of the rules of P is just about all there is to knowledge of the truth or falsehood of p. (131)*

- While this may sound like a promising epistemic thesis, consider Gödel's incompleteness theorem: for deductive systems which include a certain amount of arithmetic, there exist sentences in the system's language which cannot be decided by its rules.

$\Rightarrow$ mathematicians often embed what they want to study in richer structures, so it seems as if no mathematical theory is as self-contained as Carnap thinks they are

- But Carnap has an answer readily available (not mentioned by Shapiro): he can deny that there are cross-framework truths about the identity of propositions, i.e., if we embed a theory in a richer structure, we thereby change it.

# Characterizing logicism

William Demopoulos and Peter Clark. The logicism of Frege, Dedekind, and Russell. In S. Shapiro (ed.), *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford University Press, 2005): 129-165.

Three questions: (quoted from p. 130)

1. What is the basis for our knowledge of the infinity of the numbers?
2. How is arithmetic applicable to the world?
3. Why is reasoning by induction justified?

## Characterization (Logicism, again)

*What unifies logicists is "their opposition to the Kantian thesis that reflection on our reasoning with mere concepts... can never succeed in providing us with satisfactory answers to these three questions... [and in their] contention that the basic truths of arithmetic are susceptible of a justification that shows them to be more general than any truth secured on the basis of an intuition given a priori." (130)*

# Returning to Frege: his logic and theory of classes

- The *Begriffsschrift* (**Bs**) implicitly relies on a principle of substitution according to which for every condition $\Phi(x)$ which can be formulated in the language of **Bs**, there exists a corresponding property $P$.
- In second-order logic of a language containing a single non-logical binary relation symbol (for membership), the Basic Law V implies the

**Naive Comprehension Axiom**

$$\forall P \exists z \forall x (x \in z \leftrightarrow Px).$$

... from which the Russell-Zermelo paradox immediately follows.

Demopoulos and Clark:

> *[W]hile Frege's development of second-order logic is perfectly consistent, its elaboration, in [*Grundgesetze *(***Gg***)], to include a theory of concepts and their extension, foundered on the Russell-Zermelo paradox. However, both the mathematical development of Frege's theory of the natural numbers, and a significant component of his philosophy of mathematics, may be rendered completely independently of his theory of extensions. (134)*

# Frege's analysis of natural numbers

## Frege's "fundamental thought" (**Gg**, ix)

"A statement of number involves the predication of a concept of another concept; numerical concepts are concepts of 'second level,' which is to say, concepts under which concepts (of the first level) are said to fall." (134)

- Frege introduced a 'cardinality operator' $N$ via a contextual 'definition' ('Hume's principle'):

$$NxFx = NxGx \leftrightarrow F \approx G,$$

  i.e., the number of $F$s is the same as the number of $G$s just in case $F$s and $G$s can be brought into a one-one correspondence.

$\Rightarrow$ applicability of mathematics (here: of cardinal numbers): "arithmetic applicable to reality because the concepts under which things fall, fall under numerical concepts." (134)

# Arithmetic and reality

- In second-order logic, it can then be shown that

$$\exists^n xFx \leftrightarrow n = NxFx,$$

  i.e., the concept $F$ "falls under the numerical property expressed by the numerically definite quantifier $\exists^n x$ if and only if the Frege number of $F$ is $n$, where $n$ is defined in terms of the cardinality operator" (134)
- E.g., the fact that the number of beers in the fridge is seven is interderivable with the fact that there are seven beers in the fridge.
- Frege then goes on the deduce Hume's principle from his inconsistent theory of concepts and their extensions, but we already went over this.

- Essentially, arithmetic applies to the world because it applies to everything that can be thought, at least insofar as we can count it.
- The theorem stated on the previous slide connects a fact regarding a mathematical object (fact that the number of $F$s is $n$) with a fact that doesn't involve mathematical objects (fact that there are $n$ $F$s), i.e., connects the 'mathematical fact' that the number of beers in the fridge is seven with the 'physical fact' that there are seven beers in the fridge
- But when we leave pure arithmetic and think about applicability, we are faced with the Caesar problem again: Hume's principle does not suffice to settle the truth conditions for propositions of the form '$NxFx = n$' ($n$ not of the form $NxGx$ for some $G$).
- Demopoulos and Clark: "any language for which the problem of the application of arithmetic can be nonvacuously posed is one in which the Julius Caesar problem must also be addressed." (138)

# Final assessment of logicism

- It is possible to think that logicism was a failure because of (Zermelo-)Russell paradox and the fact that it had to rely on non-logical principles (even if second-order logic is logic, "basic laws of arithmetic are not deducible by logical means alone from the truths of logic", 161)

$\Rightarrow$ our knowledge of infinity of numbers cannot be merely logical knowledge. However,

> *[this] does absolutely nothing to detract from the depth of the logicist analysis of number. Frege's basic idea gave us quantification theory and the notion of the ancestral; Dedekind's methods provided a general mathematical technique, based upon an informal notion of set... which... became extended to the notions of ideal and lattice...; and with its development in [*Principia Mathematica*], Russell showed just what a powerful method type theory really was. (ibid.)*

*The monumental achievement of the work was to exhibit such principles and to actually show how mathematics—not merely elementary arithmetic—might be reconstructed from them. (161)... We owe to logicism first the completion of the revolution in rigor which was so important a part of nineteenth-century mathematics; we owe to it second the discovery that many arithmetical concepts are purely logical concepts; and we owe to it third the most sustained analysis of the relation between thought in general and mathematics in particular that has ever been provided. (162)*